[高等量子理论专题系列讲座]

近代量子理论中的几个数学专题

提 纲

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[第 3 讲]

泛函(路径)积分的数学分析

提 纲

一，泛函积分的分部积分运算

二，*Gauss* 型泛函积分计算举例

三，泛函 $\delta-$函数

四，泛函 *Fourier* 变换

五，泛函积分的变数变换与泛函 *Jacobi*

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一，泛函积分的分部积分运算

设算符$\hat{K}$是厄米的，且其中质量有一小的负虚部，于是它将无奇性，可以有逆算符。$\hat{K}$的定义是，若为积分形式，则为

$$\begin{gathered}\hat{K}:\ \varphi \to \psi = \hat{K}\varphi \\ \psi(x) = \hat{K}\varphi(x) = \int dy\, K(x,y)\varphi(y)\end{gathered} \tag{3.1}$$

这里$K(x,y)$是$\hat{K}$在函数空间的表示。而$K^{-1}(x,y)$则是逆算符$\hat{K}^{-1}$在函数空间的表示，是其逆变换的积分核；如是微分形式，可设$K^{-1}(x,y)$为算符$\hat{K}(x)$的 ***Green*** 函数，它满足如下方程，

$$\hat{K}(x)K^{-1}(x,y) = \delta(x-y) \tag{3.2}$$

显然，如果$\hat{K}(x)$是某个场方程左边的微分算符，则$K^{-1}(x,y) = K^{-1}(x-y)$就是该场的 ***Feynman*** 传播子（至多差一常数相因子）。可以直接解出这个 ***Green*** 函数为

$$K^{-1}(x,y) = \frac{1}{\hat{K}(x)}\delta(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \frac{1}{\hat{K}\left(\partial_\mu^{(x)} \to ip_\mu\right)} \tag{3.3}$$

1，分部积分定理

[定理] ***若泛函积分*** $I = \int D\varphi\, F[\varphi]\, G[\varphi]$ ***存在，并满足***

$$\int D\varphi \frac{\delta\left(F[\varphi]G[\varphi]\right)}{\delta\varphi(x)} = 0$$

则有分部积分公式如下

$$\boxed{\int D\varphi \frac{\delta F[\varphi]}{\delta\varphi(x)} G[\varphi] = -\int D\varphi\, F[\varphi] \frac{\delta G[\varphi]}{\delta\varphi(x)}} \tag{3.4}$$

证明：

$$\text{左边} = \int D\varphi \left\{ \frac{\delta\left(F[\varphi]G[\varphi]\right)}{\delta\varphi(x)} - F[\varphi]\frac{\delta G[\varphi]}{\delta\varphi(x)} \right\} = -\int D\varphi\, F[\varphi]\frac{\delta G[\varphi]}{\delta\varphi(x)}$$

2, 分部积分计算举例。设有如下含外源的泛函积分

$$I[\eta]=\int D\varphi \exp\left\{i\int dx'\varphi(x')\eta(x')\right\}\frac{\delta}{\delta\varphi(x)}\exp\left\{\frac{i}{2}\int dx'dy'\varphi(x')K(x'-y')\varphi(y')\right\}$$

可证有如下分部积分公式

$$\begin{aligned}&\int D\varphi \exp\left\{i\int dx'\varphi(x')\eta(x')\right\}\left(\frac{\delta}{\delta\varphi(x)}\exp\left\{\frac{i}{2}\int dx'dy'\varphi(x')K(x'-y')\varphi(y')\right\}\right)\\&=-\int D\varphi \exp\left\{\frac{i}{2}\int dx'dy'\varphi(x')K(x'-y')\varphi(y')\right\}\left(\frac{\delta}{\delta\varphi(x)}\exp\left\{i\int dx'\varphi(x')\eta(x')\right\}\right)\end{aligned} \tag{3.5}$$

证明：现往证

$$\int D\varphi\frac{\delta}{\delta\varphi(x)}\exp\left\{\frac{i}{2}\int dx'dy'\varphi(x')K(x'-y')\varphi(y')+i\int dx'\varphi(x')\eta(x')\right\}=0$$

为此，引入记号 $Z[\eta]$ 并利用后面证明的公式（**3.9b**）式:

$$\begin{aligned}Z[\eta]&\equiv\int D\varphi \exp\left\{\frac{i}{2}\int dx'dy'\varphi(x')K(x'-y')\varphi(y')+i\int dx'\varphi(x')\eta(x')\right\}\\&=\frac{1}{\sqrt{Det\,\hat{K}}}\exp\left\{\frac{-i}{2}\int dzdz'\eta(z')K^{-1}(z'-z)\eta(z)\right\}\end{aligned} \tag{3.6}$$

于是

$$\int D\varphi\frac{\delta}{\delta\varphi(x)}\exp\left\{\frac{i}{2}\int dx'dy'\varphi(x')K(x'-y')\varphi(y')+i\int dx'\varphi(x')\eta(x')\right\}$$

$$=\int D\varphi\left\{i\int dy'K(x-y')\varphi(y')+i\eta(x)\right\}\exp\left\{\frac{i}{2}\int dx'dy'\varphi K\varphi+i\int dx'\varphi\eta\right\}$$

$$=i\int dy'K(x-y')\int D\varphi\,\varphi(y')\exp\left\{\frac{i}{2}\int dx'dy'\varphi(x')K(x'-y')\varphi(y')+i\int dx'\varphi\eta\right\}$$
$$+i\eta(x)Z[\eta]$$

$$=i\int dy'K(x-y')\frac{\delta Z[\eta]}{i\delta\eta(y')}+i\eta(x)Z[\eta]$$

$$=\int dy'K(x-y')\frac{\delta}{\delta\eta(y')}\frac{1}{\sqrt{Det\hat{K}}}\exp\left\{\frac{-i}{2}\int dzdz'\eta(z')K^{-1}(z'-z)\eta(z)\right\}+i\eta(x)Z[\eta]$$

$$=\frac{-i}{\sqrt{Det\hat{K}}}\int dy'dzK(x-y')K^{-1}(y'-z)\eta(z)\exp\left\{\frac{-i}{2}\int dzdz'\eta K^{-1}\eta\right\}+i\eta(x)Z[\eta]$$

$$=-i\int dy'dzK(x-y')K^{-1}(y'-z)\eta(z)Z[\eta]+i\eta(x)Z[\eta]$$

$$=-i\eta(x)Z[\eta]+i\eta(x)Z[\eta]=0$$ 证毕。

注意，此处分部积分公式（3.5）右边将为

$$-\int D\varphi\exp\left\{\frac{i}{2}\int dx'dy'\varphi(x')K(x'-y')\varphi(y')\right\}\left(\frac{\delta}{\delta\varphi(x)}\exp\left\{i\int dx'\varphi(x')\eta(x')\right\}\right)$$

$$=-i\eta(x)\int D\varphi\exp\left\{\frac{i}{2}\int dx'dy'\varphi(x')K(x'-y')\varphi(y')+i\int dx'\varphi(x')\eta(x')\right\}$$

$$=\frac{-i\eta(x)}{\sqrt{Det\hat{K}}}\exp\left\{\frac{-i}{2}\int dx'dy'\eta(x')K^{-1}(x'-y')\eta(y')\right\} \qquad (3.7)$$

二，*Gauss* 型泛函积分计算举例

1，作为预备，下面列举几个 *Gauss* 型多重积分等式。

《多实变数情况》（*A* 为正定对称矩阵）：

$$\begin{cases}\int_{-\infty}^{\infty}\exp\left(-A_{ij}x_ix_j\right)\prod_{i=1}^{n}dx_i=\sqrt{\dfrac{\pi^n}{\det A}}\\ \int_{-\infty}^{\infty}\exp\left(-A_{ij}x_ix_j+\alpha_ix_i\right)\prod_{i=1}^{n}dx_i=\sqrt{\dfrac{\pi^n}{\det A}}\exp\left(\dfrac{1}{4}\tilde{\alpha}A^{-1}\alpha\right)\end{cases}$$

《多复变数情况》（*A* 的实部为正定对称矩阵）：

$$\begin{cases}\int_{-\infty}^{\infty}\exp\left(-Z^+AZ+\beta^+Z+Z^+\beta\right)\prod_{i=1}^{n}\dfrac{d^2z_i}{\pi}=\dfrac{1}{\det A}\exp\left(\beta^+A^{-1}\beta\right),\quad d^2z_i=dx_idy_i\\ \int_{-\infty}^{\infty}\exp\left(-Z^+AZ-\dfrac{1}{2}\tilde{Z}BZ-\dfrac{1}{2}Z^+CZ^*+\alpha^+Z+Z^+\beta\right)\prod_{i=1}^{n}\dfrac{d^2z_i}{\pi}=\\ \qquad=\left[\det\begin{pmatrix}B&\tilde{A}\\A&C\end{pmatrix}\right]\exp\left\{\dfrac{1}{2}\left(\alpha^+,\tilde{\beta}\right)\begin{pmatrix}B&\tilde{A}\\A&C\end{pmatrix}^{-1}\begin{pmatrix}\alpha^*\\ \beta\end{pmatrix}\right\},\quad \tilde{B}=B,\tilde{C}=C\end{cases}$$

《多 *Grassmann* 数情况》：

$$\int \exp\left(-\frac{1}{2}\tilde{\tau}A\tau\right)\prod_{i=1}^{n} d\tau_i = \sqrt{\det A}\,,\quad \tilde{A}=-A\,,\quad \prod_{i=1}^{n} d\tau_i = d\tau_1\cdots d\tau_n$$

$$\int \exp\left(-\frac{1}{2}\tilde{\tau}A\tau+\tilde{\alpha}\tau\right)\prod_{i=1}^{n} d\tau_i = \sqrt{\det A}\exp\left(-\frac{1}{2}\tilde{\alpha}A^{-1}\alpha\right)$$

$$\int \exp\left(-\tau^{+}A\tau+\sigma^{+}\tau+\tau^{+}\sigma\right)\prod_{i=1}^{n} d\tau_i d\tau_i^{*} = \det A\exp\left(\sigma^{+}A^{-1}\sigma\right)$$

$$\int \exp\left(-\tau^{+}A\tau-\frac{1}{2}\tilde{\tau}B\tau-\frac{1}{2}\tau^{+}C\tilde{\tau}^{+}+\tilde{\alpha}\tau+\tau^{+}\beta\right)\prod_{i=1}^{n} d\tau_i d\tau_i^{*} =$$

$$=\left[\det\begin{pmatrix} B & -\tilde{A} \\ A & C\end{pmatrix}\right]^{1/2}\exp\left\{-\frac{1}{2}\left(\tilde{\alpha},-\tilde{\beta}\right)\begin{pmatrix} B & -\tilde{A} \\ A & C\end{pmatrix}^{-1}\begin{pmatrix}\alpha \\ -\beta\end{pmatrix}\right\}$$

这里只证明第一个公式。

证明： 左边平方，为

$$\int \exp\left\{\frac{-1}{2}\left(\tilde{\tau}A\tau+\tilde{\sigma}A\sigma\right)\right\}\prod_{i=1}^{n} d\tau_i d\sigma_i = \int \frac{1}{n!}\left(\frac{-1}{2}\right)^{n}\left\{\sum_{i,j=1}^{n} A_{ij}\left(\tau_i\tau_j+\sigma_i\sigma_j\right)\right\}^{n}\prod_{i=1}^{n} d\tau_i d\sigma_i$$

$$=\int \frac{1}{n!}\left\{\sum_{i\ge j=1}^{n} A_{ji}\left(\tau_i\tau_j+\sigma_i\sigma_j\right)\right\}^{n}\prod_{i=1}^{n} d\tau_i d\sigma_i$$

$$=\int \frac{1}{n!}n!(\det A)\tau_n\tau_{n-1}\cdots\tau_1\sigma_n\cdots\sigma_1 d\tau_1 d\tau_2\cdots d\tau_n d\sigma_1 d\sigma_2\cdots d\sigma_n$$

$$=(\det A)\int \tau_n\tau_{n-1}\cdots\tau_1 d\tau_1 d\tau_2\cdots d\tau_n\sigma_n\cdots\sigma_1 d\sigma_1 d\sigma_2\cdots d\sigma_n = \det A$$

注意 *Grassmann* 数积分满足 $\int d\tau = 0,\ \int \tau_i d\tau_j = \delta_{ij}$。还用到 $\tilde{A}=-A$ 是反称矩阵，而且奇数阶反称矩阵的行列式为零。

2，实标场 *Gauss* 型二次齐次型泛函积分（$x=\left(\vec{x},x_4=ict\right)$）

$$\boxed{\int D\varphi(x)\,\exp\left\{-\frac{1}{2}\int dxdy\,\varphi(x)\hat{K}(x,y)\varphi(y)\right\}=\frac{1}{\sqrt{Det\hat{K}}}} \qquad (3.8a)$$

这里测度 $D\varphi$ 的选取保证积分等式成立。

证明：从有限维对角看，这里指数上非对角形式的双重求和 $\int dxdy\,\varphi(x)K(x,y)\varphi(y)$ 与对角形式的单重求和 $\int dx\,\varphi(x)K(x,x)\varphi(x)$ 只相差一个对 $\varphi(x)$ 的矩阵变换，或者说对中间矩阵 $\hat{K}$ 的相似变换 C。两者积分至多相差一个和行列式 $detC$ 有关的矩阵变换 *Jacobi*，

将来和分母上同样因子相消。这由（3.8）式结果行列式 $det\hat{K}=det\left(C^{-1}\hat{K}C\right)$ 也可理解。于是只需要证明对角形式即可，

$$\begin{aligned}&\int D\varphi(x)\exp\left\{-\frac{1}{2}\int dx\,\varphi(x)\hat{K}(x,x)\varphi(x)\right\}=\\&=\int D\varphi(x)\exp\left\{-\frac{1}{2}\int dx\,\varphi(x)\hat{K}\varphi(x)\right\}=\frac{1}{\sqrt{Det\hat{K}}}\end{aligned}\qquad(3.8b)$$

将它看成是有限维对角情况向连续无限维的推广。按直接计算办法，

$$左=\lim_{\substack{\Delta x\to 0\\ n\to\infty}}\int\prod_{i=1}^{n}\frac{d\varphi(x_i)}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\Delta x\cdot\varphi(x_i)K(x_i,x_i)\cdot\varphi(x_i)\right\}$$

$$=\lim_{\substack{\Delta x\to 0\\ n\to\infty}}\int\prod_{i=1}^{n}\frac{d\varphi_i}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\varphi_i\left(K_{ii}\Delta x\right)\cdot\varphi_i\right\}=\frac{1}{\sqrt{Det\hat{K}}}$$

一般而言，在 x 取值分立的有限维情况下，普通函数 $K(x)$ 对其后函数 $\varphi(x)$ 的作用相当于对角矩阵；算符 $\hat{K}$ 对其后函数 $\varphi(x)$ 的作用相当于非对角矩阵；而在函数空间中，算符 $\hat{K}$ 的作用可视作 $\hat{K}\varphi(x)=\int dyK(x,y)\varphi(y)$。

3，实标场 *Gauss* 型非齐次二次型泛函积分

$$\boxed{\begin{aligned}&\int D\varphi(x)\exp\left\{\int dx\exp\left(-\frac{1}{2}\varphi(x)\hat{K}\varphi(x)+J(x)\varphi(x)\right)\right\}\\&=\frac{1}{\sqrt{Det\hat{K}}}\exp\left\{\frac{1}{2}\int d(xy)J(x)K^{-1}(x,y)J(y)\right\}\end{aligned}}\qquad(3.9a)$$

测度 $D\varphi$ 的选取应保证此积分在 $J=0$ 时回归到等式（3.8b）。

证明：这也是有限模情况向连续模情况的推广。作函数平移变换（相应的泛函 *Jacobi* 等于 1，说明见结尾处）：

$$\varphi(x)\to\Phi(x)=\varphi(x)-\int d^4x'K^{-1}(x,x')J(x')$$

于是有

$$\frac{-1}{2}\int d^4x\,\Phi(x)\,\hat{K}\,\Phi(x)=$$

$$=\frac{-1}{2}\int d^4x\left[\varphi(x)-\int d^4x'K^{-1}(x,x')J(x')\right]\hat{K}\left[\varphi(x)-\int d^4x''K^{-1}(x,x'')J(x'')\right]$$

$$\begin{aligned}=\frac{-1}{2}&\int d^4x\varphi(x)\hat{K}\varphi(x)+\frac{1}{2}\int d^4(xx')K^{-1}(x,x')J(x')\hat{K}\varphi(x)\\&+\frac{1}{2}\int d^4(xx'')\varphi(x)\hat{K}K^{-1}(x,x'')J(x'')\\&-\frac{1}{2}\int d^4(xx'x'')K^{-1}(x,x')J(x')\hat{K}K^{-1}(x,x'')J(x'')\end{aligned}$$

$$\begin{aligned}=\frac{-1}{2}&\int d^4x\,\varphi(x)\hat{K}\varphi(x)+\frac{1}{2}\int d^4(xx')K^{-1}(x,x')J(x')\hat{K}\varphi(x)\\&+\frac{1}{2}\int d^4x\,\varphi(x)J(x)-\frac{1}{2}\int d^4(xx')K^{-1}(x,x')J(x')J(x)\end{aligned}$$

如果$\hat{K}(x)$中含$\partial_\mu^{(x)}$二阶偏导数，则第二项经分部积分两次，从而可知它和第三项相等。再将第四项移至等式左边，即得下面等式：

$$\frac{-1}{2}\int d^4x\Phi(x)\hat{K}\Phi(x)+\frac{1}{2}\int d^4(xy)J(x)K^{-1}(x,y)J(y)=-\frac{1}{2}\int d^4x\,\varphi\hat{K}\varphi+\int d^4x\varphi J$$

此式右边即为（**3.9a**）式左边的指数，再利用（**3.8b**）式，即得

$$\int D\Phi\exp\left\{\frac{-1}{2}\int d^4x\,\Phi(x)\hat{K}\Phi(x)+\frac{1}{2}\int d^4(xy)J(x)K^{-1}(x,y)J(y)\right\}$$

$$=\frac{1}{\sqrt{Det\hat{K}}}\exp\left\{\frac{1}{2}\int d^4(xy)J(x)K^{-1}(x,y)J(y)\right\}$$

证毕。

最后，补充说明此变换 ***Jacobi*** 为 **1**。即便对多分量实标场，由于

$$\frac{\delta\varphi_i(x)}{\delta\Phi_j(y)}=\frac{\delta}{\delta\Phi_j(y)}\left[\Phi_i(x)+\int d^4x'K^{-1}_{ij}(x,x')J_j(x')\right]=\delta_{ij}\delta(x-y)=\delta_{ij}\langle x|\hat{I}|y\rangle$$

于是$J\left(\frac{\delta\varphi_i}{\delta\Phi_j}\right)=\det\hat{I}=\exp\left\{tr\ln\hat{I}\right\}=1$。（**3.9a**）式也常常写作：

$$\left(-K(x,y)\to iK(x,y)\ ;K^{-1}(x,y)\to iK^{-1}(x,y);J(x)\to iJ(x)\right)$$

$$\int D\varphi(x)\exp\left\{\frac{i}{2}\int d(xy)\varphi(x)K(x,y)\varphi(y)+i\int dx\,\varphi(x)J(x)\right\}$$
$$=\frac{1}{\sqrt{Det\hat{K}}}\exp\left\{\frac{-i}{2}\int d(xy)J(x)K^{-1}(x,y)J(y)\right\} \quad (3.9b)$$

此式的具体算例是：有质量的矢量玻色子场。即令

$$\hat{K}_{\mu\nu}(x)=\left[\delta_{\mu\nu}\left(\partial_\mu^{(x)2}-M^2\right)-\partial_\mu^{(x)}\partial_\nu^{(x)}\right],\quad M^2\to M^2(1-i\varepsilon)$$

$$\int\prod_\mu DB_\mu\exp\left\{i\int d^4x\left[\frac{1}{2}B_\mu(x)\hat{K}_{\mu\nu}(x)B_\nu(x)+J_\mu(x)B_\mu(x)\right]\right\}$$
$$=\frac{1}{\sqrt{\left|Det\hat{K}_{\mu\nu}\right|}}\exp\left\{-\frac{1}{2}\int d^4(xy)J_\mu(x)D_F(x,y)_{\mu\nu}J_\nu(y)\right\} \quad (3.9c)$$

（3.9c）式的测度为 $DB_\mu=D\left(\frac{B_\mu}{\sqrt{2\pi i}}\right)$。同时有

$$\hat{K}_{\mu\nu}D_F(x,y)_{\nu\lambda}=i\delta_{\mu\lambda}\delta(x-y),\quad D_F(x,y)_{\nu\lambda}=-i\int\frac{d^4k}{(2\pi)^4}e^{ik(x-y)}\frac{\left(\delta_{\nu\lambda}+k_\nu k_\lambda/k^2\right)}{\sqrt{Det\hat{K}_{\mu\nu}}}$$

$$\hat{K}_{\mu\nu}\left(\hat{K}\right)^{-1}_{\nu\lambda}(x,y)=\delta_{\mu\lambda}\delta(x-y)$$

4，复标场情况

$$\int D\varphi D\varphi^*\exp\left\{-\int dxdy\left[\varphi^+(x)A(x,y)\varphi(y)+\frac{1}{2}\tilde{\varphi}(x)B(x,y)\varphi(y)\right.\right.$$
$$\left.\left.+\frac{1}{2}\varphi^+(x)C(x,y)\tilde{\varphi}^+(y)\right]+\int dx\left[J^+(x)\varphi(x)+\varphi^+(x)J(x)\right]\right\}$$

$$=\left[\det\begin{pmatrix}\hat{B} & \tilde{\hat{A}}\\ \hat{A} & \hat{C}\end{pmatrix}\right]^{-1/2}\exp\left\{\frac{1}{2}\int dxdy\begin{pmatrix}J^+(x) & \tilde{J}(x)\end{pmatrix}\begin{pmatrix}B(x,y) & \tilde{A}(x,y)\\ A(x,y) & C(x,y)\end{pmatrix}^{-1}\begin{pmatrix}\tilde{J}^+(y)\\ J(y)\end{pmatrix}\right\}$$

（3.10a）

这里测度为 $D\varphi D\varphi^*=\prod_x\frac{(\Delta x)^4}{2\pi}d\varphi d\varphi^*$， $\tilde{A}(x,y)=A(y,x)$。

证明： （3.10a）式是有限模情况向连续模情况的推广。左边为

$$\int D\varphi D\varphi^* \exp\left\{-\frac{1}{2}\int dxdy\begin{pmatrix}\tilde{\varphi}(x) & \varphi^+(x)\end{pmatrix}\begin{pmatrix}B(x,y) & A(y,x)\\ A(x,y) & C(x,y)\end{pmatrix}\begin{pmatrix}\varphi(y)\\ \tilde{\varphi}^+(y)\end{pmatrix}\right.$$
$$\left.+\int dx\begin{pmatrix}J^+(x) & \tilde{J}(x)\end{pmatrix}\begin{pmatrix}\varphi(y)\\ \tilde{\varphi}^+(y)\end{pmatrix}\right\}$$

利用（3.9b）式，得

$$左=\left(\left|\det\begin{pmatrix}B & \tilde{A}\\ A & C\end{pmatrix}\right|\right)^{-1/2}\exp\left\{\frac{1}{2}\int dxdy\begin{pmatrix}J^+(x) & \tilde{J}(x)\end{pmatrix}\begin{pmatrix}B & \tilde{A}\\ A & C\end{pmatrix}^{-1}(x,y)\begin{pmatrix}\tilde{J}^+(y)\\ J(y)\end{pmatrix}\right\}$$

证毕。

（3.10a）式具体算例是复标量场：$\hat{A}(x)=-i\left(\partial_\mu^{(x)2}-m^2\right)$，$m^2\to m^2-i\varepsilon$，

$$\int D\varphi D\varphi^* \exp\left\{i\int d^4x\left[\varphi^*\partial_\mu^2\varphi-m^2\varphi^*\varphi+J^*\varphi+J\varphi^*\right]\right\}= \\ =\left|Det\left(\partial_\mu^2-m^2\right)\right|^{-1}\exp\left\{-\int d^4(xy)J^*(x)\Delta_F(x,y)J(y)\right\} \qquad (3.10b)$$

此处测度为 $D\varphi D\varphi^*=\frac{D\varphi}{\sqrt{\pi i}}\frac{D\varphi^*}{\sqrt{\pi i}}$，并且有

$$\hat{A}(x)A^{-1}(x,y)=\delta(x-y)\to A^{-1}(x,y)=\Delta_F(x,y)$$

5，*Gauss* 型二次齐次 *Grassmann* 数泛函积分

$$\int D\psi D\bar{\psi}\exp\left\{-\int dx\,\bar{\psi}K\psi\right\}=DetK \qquad (3.11)$$

证明：这是向连续无穷自由度的推广，由相应的 *Grassmann* 数第 3 个公式的齐次情况推知。

6，*Gauss* 型二次非齐次 *Grassmann* 数泛函积分

设 $K(x)$ 为普通函数，可证有

$$\int D\psi D\bar{\psi}\exp\left\{\int dx\left[-\bar{\psi}K\psi+\bar{\eta}(x)\psi(x)+\bar{\psi}(x)\eta(x)\right]\right\}= \\ =\left|DetK\right|\exp\left\{\int dx\,\bar{\eta}(x)K^{-1}(x,x)\eta(x)\right\} \qquad (3.12a)$$

或

$$\boxed{\begin{aligned}&\int D\psi D\bar{\psi}\exp\left\{i\int dx\left[-\bar{\psi}K\psi+\bar{\eta}(x)\psi(x)+\bar{\psi}(x)\eta(x)\right]\right\}=\\&=|DetK|\exp\left\{-i\int dx\,\bar{\eta}(x)K^{-1}(x,x)\eta(x)\right\}\end{aligned}} \qquad (3.12b)$$

证明：在函数空间中（取定一组正交完备基，将ψ和$\bar{\psi}$展开，知这时是对角矩阵情况），普通函数$K(x)$对应的矩阵是对角的，于是直接使用有限模情况公式即可。

对于$K(x)$是算符的情况，有

$$\boxed{\begin{aligned}&\int D\psi D\bar{\psi}\exp\left\{i\int dx\left[-\bar{\psi}\hat{K}\psi+\bar{\eta}(x)\psi(x)+\bar{\psi}(x)\eta(x)\right]\right\}=\\&=|Det\hat{K}|\exp\left\{-i\int dxdy\,\bar{\eta}(x)K^{-1}(x,y)\eta(y)\right\}\end{aligned}} \qquad (3.12c)$$

此 式 具 体 算 例 是 旋 量 场 ： $\hat{K}(x)=-\left(\gamma_\mu\partial_\mu^{(x)}+\kappa\right)$, $\kappa=m-i\varepsilon$ 。

$\hat{K}^{-1}=\hat{K}^{-1}(x,y)$的定义是：用它右乘$\hat{K}(x)$等于$\delta(x-y)$，即

$$\hat{K}(x)\hat{K}^{-1}(x,y)=-\left(\gamma_\mu\partial_\mu^{(x)}+\kappa\right)K^{-1}(x,y)=\delta(x-y)$$

$$\therefore\quad K^{-1}(x,y)=\frac{1}{\hat{K}(x)}\delta(x-y)=\int\frac{d^4p}{(2\pi)^4}e^{ip(x-y)}\frac{-1}{i\not{p}+m(1-i\varepsilon)}=-iS_F(x,y)$$

证明：这时在函数空间中，$\hat{K}$对其后$\psi(x)$的作用相当于非对角矩阵。此时$\hat{K}^{-1}$定义见前面叙述。证明也直接用有限模公式，

左边=

$$=\int D\psi D\bar{\psi}\exp\left\{\int d^4x\left[-\bar{\psi}(x)\left(-i\hat{K}(x)\right)\psi(x)+\left(i\bar{\eta}(x)\right)\psi(x)+\bar{\psi}(x)\left(i\eta(x)\right)\right]\right\}$$

$$=\left|Det\left(-i\hat{K}(x)\right)\right|\exp\left\{\int d^4(xy)\left(i\bar{\eta}(x)\right)\left(-i\hat{K}(x)\right)^{-1}\left(i\eta(y)\right)\right\}$$

$$=\left|Det\hat{K}(x)\right|\exp\left\{-i\int d^4(xy)\bar{\eta}(x)\left(\hat{K}(x)\right)^{-1}\eta(y)\right\}$$

三，泛函$\delta-$函数

1，泛函$\delta-$函数的两种表达式

i，第一种表达式

$$\boxed{\delta[\varphi]=\prod_x \delta(\varphi(x))=\int[Df(x)]\exp\left\{i\int d^4x'\varphi(x')f(x')\right\}} \qquad （3.13a）$$

或

$$\boxed{\delta[\varphi-\psi]=\prod_x \delta(\varphi(x)-\psi(x))=\int[Df(x)]\exp\left\{i\int d^4x'(\varphi(x')-\psi(x'))f(x')\right\}}$$

（3.13b）

并有

$$\boxed{\int D\varphi F[\varphi]\delta[\varphi-\psi]=F[\psi]} \qquad （3.14）$$

对 *Gauss* 型泛函 $F[\varphi]$ 情况，（3.14）式可以直接证明如下：

证明： 设泛函 $F[\varphi]=\exp\left\{\frac{i}{2}\int d^4(xy)\varphi(x)K(x-y)\varphi(y)\right\}$，将此泛函乘以泛函 $\delta-$ 函数并作泛函积分，为

左边=

$$=\int D\varphi\exp\left\{\frac{i}{2}\int d^4(xy)\varphi(x)K(x-y)\varphi(y)\right\}\int Df\exp\left\{i\int d^4x[\varphi(x)-\psi(x)]f(x)\right\}$$

$$=\int Df\exp\left\{-i\int d^4x\psi(x)f(x)\right\}\int D\varphi\exp\left\{\frac{i}{2}\int d^4(xy)\varphi(x)K(x-y)\varphi(y)+\right.$$
$$\left.+i\int d^4x\varphi(x)f(x)\right\}$$

$$=\int Df\exp\left\{-i\int d^4x\psi(x)f(x)\right\}\frac{1}{\sqrt{|Det\hat{K}|}}\exp\left\{-\frac{i}{2}\int d^4(xy)f(x)K^{-1}(x-y)f(y)\right\}$$

$$=\frac{1}{\sqrt{|Det\hat{K}|}}\int Df\exp\left\{\frac{i}{2}\int d^4(xy)f(x)(-K^{-1}(x-y))f(y)+i\int d^4x(-\psi(x))f(x)\right\}$$

$$=\frac{1}{\sqrt{|Det\hat{K}|}}\frac{1}{\sqrt{|Det(-\hat{K}^{-1})|}}\exp\left\{\frac{i}{2}\int d^4(xy)\psi(x)K(x-y)\psi(y)\right\}$$

$$=\exp\left\{\frac{i}{2}\int d^4(xy)\psi(x)K(x-y)\psi(y)\right\}=F[\psi]$$ 证毕。

由此处泛函 $\delta-$ 函数定义（3.13b）式可知，泛函 $\delta-$ 函数为偶函数。

ii, **第二种表达式**

$$\boxed{\delta\left[F_\alpha[\varphi]\right]\equiv\prod_{x,\alpha}\delta\left(F_\alpha[\varphi]\right)=\lim_{\varepsilon\to 0}\exp\left\{-\frac{i}{2\varepsilon}\sum_\alpha\int d^4x\left(F_\alpha[\varphi]\right)^2\right\}}\qquad(3.15)$$

此公式将 *Gauss* 型泛函$\delta-$函数表示推广成为 *Fresnel* 型的表示。若分立指标α为指定的，则右边指数上没有对α求和。

证明： 首先将普通$\delta-$函数的 *Gauss* 型表示推广到 *Fresnel* 型表示；

$$\delta(x)=\lim_{b\to 0}\frac{1}{\sqrt{\pi b}}e^{-x^2/b}\to\delta(x)=\lim_{b\to 0}\frac{e^{i\pi/4}}{\sqrt{\pi b}}e^{-ix^2/b}$$

于是有

$$\begin{aligned}\delta\left[F_\alpha[\varphi]\right]\equiv\prod_{x,\alpha}\delta\left(F_\alpha[\varphi]\right)&\propto\lim_{b\to 0}\exp\left\{-\frac{i}{b}\sum_{x,\alpha}\left(F_\alpha[\varphi]\right)^2\right\}\\&=\lim_{\substack{b\to 0\\ \Delta x\to 0}}\exp\left\{-\frac{i}{b(\Delta x)^4}\sum_\alpha\int d^4x\left(F_\alpha[\varphi]\right)^2\right\}\\&=\lim_{\varepsilon\to 0}\exp\left\{\frac{-i}{2\varepsilon}\sum_\alpha\int d^4x\left(F_\alpha[\varphi]\right)^2\right\}\end{aligned}$$

这里在“$\propto$”号的一步中略去了$\delta-$函数中与函数$\varphi(x)$无关的常数因子，因为它与生成泛函分母归一化系数中的对应因子相消。

2，泛函$\delta-$函数的自变量变换

i, 泛函$\delta-$函数为偶函数。即总有

$$\boxed{\delta[\varphi-\psi]=\delta[\psi-\varphi]}\qquad(3.16)$$

由第一种表达式（3.13b）可知，这相当于对泛函积分变数做了变换$f(x)\to-f(x)$。注意，做此变换时泛函积分上下限也变换，所以积分数值不变，故有$\int\left[Df(x)\right]=\int\left[D(-f(x))\right]$。

ii, 泛函$\delta-$函数的自变量变换

$$\boxed{\delta\left[\varphi-\hat{M}\psi\right]=\left|Det\hat{M}\right|^{-1}\delta\left[\psi-\hat{M}^{-1}\varphi\right]} \qquad (3.17)$$

这里$\delta\left[\varphi-\hat{M}\psi\right]=\prod_x\delta\left(\varphi(x)-\left(\hat{M}\psi\right)(x)\right)$。

证明： 设有可逆的映射 $\hat{M}:\psi(x)\to\varphi(x)$（即 $\hat{M}\psi=\varphi$），于是 $F[\varphi]=F\left[\varphi[\psi]\right]\equiv G[\psi]$。设$\tilde{\varphi}$和$\tilde{\psi}$为满足$\hat{M}\tilde{\psi}=\tilde{\varphi}$的任意一组函数对，按泛函$\delta-$函数定义有

$$\begin{cases}\int D\varphi\cdot\delta\left[\varphi-\hat{M}\tilde{\psi}\right]F[\varphi]=F\left[\hat{M}\tilde{\psi}\right]\\ \int D\psi\cdot\delta\left[\psi-\hat{M}^{-1}\tilde{\varphi}\right]G[\psi]=G\left[\hat{M}^{-1}\tilde{\varphi}\right]\end{cases}$$

第一式的右边和左边分别等于

$$\begin{cases}F\left[\hat{M}\tilde{\psi}\right]=F[\tilde{\varphi}]=G[\tilde{\psi}]=G\left[\hat{M}^{-1}\tilde{\varphi}\right]\\ \int D\psi\left|Det\left(\frac{\delta\varphi}{\delta\psi}\right)\right|\delta\left[\varphi-\hat{M}\tilde{\psi}\right]F\left[\varphi[\psi]\right]=\int D\psi\left|Det\hat{M}\right|\delta\left[\varphi-\hat{M}\tilde{\psi}\right]G[\psi]\end{cases}$$

与第二式相比较，并略去“~”，即得

$$\delta\left[\varphi-\hat{M}\psi\right]=\left|Det\hat{M}\right|^{-1}\delta\left[\psi-\hat{M}^{-1}\varphi\right]$$

此即（3.17）式。 证毕。

于是，$\delta-$函数的前面定义式又可以改写成为：

$$\begin{aligned}\delta\left[\varphi-\hat{M}\psi\right]&=\prod_x\delta\left(\varphi(x)-\int dyM(x-y)\psi(y)\right)=\left|Det\hat{M}\right|^{-1}\delta\left[\psi-\hat{M}^{-1}\varphi\right]\\&=\left|Det\hat{M}\right|^{-1}\prod_x\delta\left(\psi(x)-\int dyM^{-1}(x-y)\varphi(y)\right)\\&=\left|Det\hat{M}\right|^{-1}\int Df\exp\left\{i\int d^4x\left[\int M^{-1}(x-y)\varphi(y)dy-\psi(x)\right]f(x)\right\}\end{aligned} \qquad (3.18)$$

这些记法逻辑上是自洽的。因为对任意泛函$F[\varphi]$，按$\delta-$函数定义：

$$\int D\varphi F[\varphi]\delta\left[\varphi-\hat{M}\psi\right]=F\left[\hat{M}\psi\right]$$

另一方面，按泛函积分变换和$\delta-$函数变换的定义，上式左边为：

$$\int D\psi \left| Det\left(\frac{\delta\varphi}{\delta\psi}\right) \right| F\left[\varphi[\psi]\right]\delta\left[\varphi-\hat{M}\psi\right]$$
$$=\int D\psi \left| Det\hat{M} \right| F\left[\varphi[\psi]\right]\delta\left[\varphi-\hat{M}\psi\right]$$
$$=\int D\psi F\left[\varphi[\psi]\right]\delta\left[\psi-\hat{M}^{-1}\varphi\right]=F\left[\varphi\left[\hat{M}^{-1}\varphi\right]\right]$$
$$=F\left[\hat{M}\hat{M}^{-1}\varphi\right]=F[\varphi]=F\left[\hat{M}\psi\right]$$

四，泛函 *Fourier* 变换

1，泛函 *Fourier* 变换定义

显然，下面 *Gauss* 型泛函积分公式，

$$\boxed{\begin{aligned} Z[\eta] &= \int D\varphi \exp\left\{\frac{i}{2}\int dxdy\,\varphi(x)K(x-y)\varphi(y)+i\int dx\,\varphi(x)\eta(x)\right\} \\ &= \frac{1}{\sqrt{Det\hat{K}}}\exp\left\{\frac{-i}{2}\int dxdy\,\eta(x)K^{-1}(x-y)\eta(y)\right\}\end{aligned}} \quad (3.19)$$

是一个**泛函 *Fourier* 变换式**：中间的指数二次型的被积泛函 $F[\varphi]$，

$$F[\varphi]=\exp\left\{\frac{i}{2}\int dxdy\,\varphi(x)K(x-y)\varphi(y)\right\}$$

是场量函数 $\varphi(x)$ 的 *Gauss* 型泛函 $F[\varphi]$，是此**泛函 *Fourier* 变换的原泛函**；右边最后积出的结果是外源函数 $\eta(x)$ 的 *Gauss* 型泛函 $Z[\eta]$，是**泛函 *Fourier* 变换的像泛函**。就是说，这两个泛函之间的关系为：

"泛函 $Z[\eta]$ 是泛函 $F[\varphi]$ 的 *Fourier* 变换。或者说，

$$\exp\left\{\frac{i}{2}\int dxdy\,\varphi(x)K(x-y)\varphi(y)\right\} \Leftrightarrow \frac{1}{\sqrt{Det\hat{K}}}\exp\left\{\frac{-i}{2}\int dxdy\,\eta(x)K^{-1}(x-y)\eta(y)\right\}$$

是泛函 *Fourier* 变换对。"

一般说，[**泛函 *Fourier* 变换定义**]：

"如果下面泛函积分 $Z[\eta]$ 存在，

$$Z[\eta]=\int D\varphi(x)F[\varphi(x)]\exp\left\{i\int dx'\,\varphi(x')\eta(x')\right\}$$

则称外源泛函 $Z[\eta]$ 为场量泛函 $F[\varphi]$ 的 *Fourier* 变换像泛函。”

更一般地，鉴于有等式，

$$\int D\varphi\,\varphi(x_1)\cdots\varphi(x_n)\exp\left\{\frac{i}{2}\int dxdy\,\varphi(x)K(x-y)\varphi(y)+i\int dx\,\varphi(x)\eta(x)\right\}$$

$$=\frac{1}{\sqrt{Det\hat{K}}}\frac{\delta}{i\delta\eta(x_1)}\cdots\frac{\delta}{i\delta\eta(x_n)}\exp\left\{\frac{-i}{2}\int dxdy\,\eta(x)K^{-1}(x-y)\eta(y)\right\} \qquad (3.20)$$

“称场量函数 $\varphi(x)$ 的 *Gauss* 型泛函

$$F[\varphi]=\varphi(x_1)\cdots\varphi(x_n)\exp\left\{\frac{i}{2}\int dxdy\,\varphi(x)K(x-y)\varphi(y)\right\}$$

为泛函 *Fourier* 变换的原泛函，而称外源 $\eta(x)$ 的 *Gauss* 型泛函

$$\frac{1}{\sqrt{Det\hat{K}}}\frac{\delta}{i\delta\eta(x_1)}\cdots\frac{\delta}{i\delta\eta(x_n)}\exp\left\{\frac{-i}{2}\int dxdy\,\eta(x)K^{-1}(x-y)\eta(y)\right\}$$

为泛函 *Fourier* 变换的像泛函。”

2，例算。作为泛函 ***Fourier*** 变换和 $\delta-$函数的一个应用，求证如下结论： 对泛函积分作变数变换

$$\begin{aligned}\varphi(x)=\hat{M}\psi(x)&=\int M(x,y)\psi(y)dy\\&=c_0(x)+\psi(x)+\int N(x,y)\psi(y)dy\\&\equiv c_0(x)+\psi(x)+N[\psi]\end{aligned}$$

这时有下述泛函积分转换

$$\boxed{\begin{aligned}&\int D\varphi\exp\left\{\frac{i}{2}\int\varphi(x)K(x-y)\varphi(y)dxdy+i\int dx\varphi(x)J(x)\right\}=\\&=\int D\psi\,Det\left(1+\frac{\delta N}{\delta\psi}\right)\exp\left\{\frac{i}{2}\int\left(\hat{M}\psi\right)(x)K(x-y)\left(\hat{M}\psi\right)(y)dxdy\right.\\&\quad\left.+i\int\left(\hat{M}\psi\right)(x)J(x)dx\right\}\end{aligned}} \qquad (3.21)$$

证明： 变换前被积函数是 ***Gauss*** 型，积分有定义。根据 ***Fourier*** 变换思想，只需证明 ***Fourier*** 变换等式（**3.21**）两边的原函数相等即可。

将等式左边看作像泛函（记作 $Z[J]$），其原泛函 $F[\varphi]$ 显然是

$$F[\varphi]=\exp\left\{\frac{i}{2}\int\varphi(x)K(x-y)\varphi(y)dxdy\right\}$$

将等式右边也看作像泛函（记作 $\tilde{Z}[J]$），其原泛函 $\tilde{F}[\psi]$ 是

$$\tilde{F}[\psi]=\int DJ\cdot\tilde{Z}[J]\cdot\exp\left\{-i\int dxJ(x)\psi(x)\right\}$$

$$=\int D\psi Det\left(1+\frac{\delta N}{\delta\psi}\right)\exp\left\{\frac{i}{2}\int\left(\hat{M}\psi\right)(x)K(x-y)\left(\hat{M}\psi\right)(y)dxdy\right\}\cdot$$
$$\cdot\int DJ\exp\left\{i\int J(x)\left[\left(\hat{M}\psi\right)(x)-\psi(x)\right]dx\right\}$$

$$=\int D\psi Det\left(1+\frac{\delta N}{\delta\psi}\right)\exp\left\{\frac{i}{2}\int\left(\hat{M}\psi\right)(x)K(x-y)\left(\hat{M}\psi\right)(y)dxdy\right\}\cdot$$
$$\cdot\delta\left[\left(\hat{M}\psi\right)(x)-\psi(x)\right]$$

$$=\int D\psi Det\left(1+\frac{\delta N}{\delta\psi}\right)\exp\left\{\frac{i}{2}\int\psi(x)K(x-y)\psi(y)dxdy\right\}\cdot$$
$$\cdot\delta\left[\left(\hat{M}\psi\right)(x)-\psi(x)\right]$$

$$=\int D\varphi\exp\left\{\frac{i}{2}\int\psi(x)K(x-y)\psi(y)dxdy\right\}\delta\left[\varphi(x)-\psi(x)\right]$$
$$=\exp\left\{\frac{i}{2}\int\varphi(x)K(x-y)\varphi(y)dxdy\right\}=F[\varphi]$$

果然（3.21）式两边的原泛函相等，所以（3.21）式是等式。证毕。

五，泛函积分的变数变换与泛函 *Jacobi*

1，泛函积分的变数变换，泛函 *Jacobi*

设有线性函数映射 $M:\varphi(x)\to\psi(x)$，新场量 $\varphi(x)$ 由老场量 $\psi(x)$ 按如下变换相联系

$$\varphi_\alpha(x)=\int M_{\alpha\beta}(x,y)\psi_\beta(y)dy\equiv\left(\hat{M}\psi\right)_\alpha(x)\qquad(3.22a)$$

α,β 为内禀、外在全部附属空间的分量指标。在此泛函变数变换下，积分测度相应变换为

$$D\varphi(x)=\prod_{\alpha,x}\frac{d\varphi_\alpha(x)}{\sqrt{C}}=\prod_{\alpha\beta,xy}\frac{\delta\varphi_\alpha(x)}{\delta\psi_\beta(y)}\cdot\frac{d\psi_\beta(y)}{\sqrt{C}}$$

$$\therefore \qquad \boxed{D\varphi(x)=J\left(\frac{\delta\varphi_\alpha(x)}{\delta\psi_\beta(y)}\right)D\psi(y)} \qquad (3.22b)$$

这里，泛函 ***Jacobi*** 计算为

$$\frac{\delta\varphi_\alpha(x)}{\delta\psi_\beta(y)}=M_{\alpha\beta}(x,y)\to\left(\frac{\delta\varphi_\alpha(x)}{\delta\psi_\beta(y)}\right)=\left(M_{\alpha\beta}(x,y)\right)\equiv M(x,y)\to$$

$$J\left(\frac{\delta\varphi_\alpha(x)}{\delta\psi_\beta(y)}\right)=\exp\left\{tr\int dxdy\delta(x-y)\ln M(x,y)\right\}=\exp\left\{S_P\left(\ln\hat{M}\right)\right\}=Det\hat{M}$$

这里，tr 是对全部附属空间脚标 α,β 求迹。于是泛函 ***Jacobi*** 为

$$\boxed{J\left(\frac{\delta\varphi_\alpha}{\delta\psi_\beta}\right)=\begin{cases}\exp\left[S_P\left(\ln\hat{M}\right)\right]=Det\hat{M}, & for\ boson\\ \exp\left[-S_P\left(\ln\hat{M}\right)\right]=\left(Det\hat{M}\right)^{-1}, & for\ \ fermion\end{cases}} \qquad (3.23)$$

此处对 ***fermion*** 的泛函积分变换，由于规定新旧 ***Grassmann*** 数积分规则保持不变（$\int d\tau=0,\ \int\tau_i d\tau_j=\delta_{ij}$），于是被积函数变换产生的 $Det\hat{M}$ 和积分测度引出的 $J\left(\hat{M}\right)$，它们乘积必须相消 $Det\hat{M}\cdot J\left(\hat{M}\right)=I$。所以 $J\left(\hat{M}\right)=\left(Det\hat{M}\right)^{-1}$。

最后，泛函积分的变数变换为

$$\int F\left[\varphi_\alpha(x)\right]\prod_\alpha D\varphi_\alpha(x)=\int F\left[\psi_\beta(y)\right]J\left(\frac{\delta\varphi_\alpha}{\delta\psi_\beta}\right)\prod_\beta D\psi_\beta(y) \qquad (3.24a)$$

或写成

$$\int D\varphi\, F[\varphi]=\int D\psi\left|Det\hat{M}\right|^{\pm1}F\left[\hat{M}\psi\right] \qquad (3.24b)$$

“±”由 ***boson***（***fermion***）决定，$\varphi(x)\sim\psi(x)$ 关系由（**3.22a**）式决定。

2，无穷小泛函变分，泛函 *Jacobi*

如果场变数的变化是无穷小的泛函变分，即变换 $\hat{M}$ 接近于恒等变换 $\hat{M}=\hat{I}+\hat{A}$，$\hat{A}$ 为含有穷小常数的算符，如同经常做的那样。这时

可对上面 ***Jacobi*** 作近似，从而写成更方便的形式：

$$\exp\left[\pm tr\left(\ln \hat{M}\right)\right]=\exp\left[\pm tr\left(\ln\left(\hat{I}+\hat{A}\right)\right)\right]\simeq\exp\left[\pm tr\hat{A}\right]\simeq\hat{I}\pm tr\hat{A}$$

可得此无穷小变换的泛函 ***Jacobi*** 为

$$\boxed{J\left(\frac{\delta\psi'_\alpha}{\delta\psi_\beta}\right)\simeq\begin{cases}\hat{I}+tr\hat{A}, & for\ boson\\ \hat{I}-tr\hat{A}, & for\ fermion\end{cases}} \quad (3.25)$$

举例：泛函 $\int D\psi D\bar{\psi}\prod_\mu DB_\mu F\left(\psi,\bar{\psi},B_\mu\right)$ 经受如下场量变换

$$\begin{cases}\psi(x)\to\psi'(x)=\left(1+\frac{3}{2}\lambda(x)\right)\psi(x)\\ \bar{\psi}(x)\to\bar{\psi}'(x)=\left(1+\frac{3}{2}\lambda(x)\right)\bar{\psi}(x)\\ B_\mu(x)\to B'_\mu(x)=\left(1+\lambda(x)\right)B_\mu(x)\end{cases} \quad (3.26a)$$

这时泛函 ***Jacobi*** 为

$$J[\lambda]=J_\psi[\lambda]J_{\bar{\psi}}[\lambda]J_B[\lambda]=\left(\hat{I}-tr\hat{A}_\psi\right)\left(\hat{I}-tr\hat{A}_{\bar{\psi}}\right)\left(\hat{I}+tr\hat{A}_B\right)$$

其中比如对 ψ，有

$$\psi'(x)=\int\delta(x-y)\left(1+\frac{3}{2}\lambda(x)\right)\psi(y)d^4y$$

$$\therefore\qquad M(x,y)\left(=\langle x|\hat{M}|y\rangle\right)=\delta(x-y)\left(1+\frac{3}{2}\lambda(x)\right)$$

于是有

$$\begin{cases}A_\psi(x,y)=\langle x|\hat{M}_\psi-1|y\rangle=\frac{3}{2}\delta(x-y)\lambda(x), & for\ \psi-4\dim\\ A_{\bar{\psi}}(x,y)=\langle x|\hat{M}_{\bar{\psi}}-1|y\rangle=\frac{3}{2}\delta(x-y)\lambda(x), & for\ \bar{\psi}-4\dim\\ A_B(x,y)=\langle x|\hat{M}_B-1|y\rangle=\delta(x-y)\lambda(x), & for\ B_\mu-4\dim\end{cases}$$

$$\begin{cases} trA_{\psi} = 4\cdot\frac{3}{2}\delta(0)\int\lambda(x)d^4x = 6\delta(0)\int\lambda(x)d^4x \\ trA_{\bar{\psi}} = 4\cdot\frac{3}{2}\delta(0)\int\lambda(x)d^4x = 6\delta(0)\int\lambda(x)d^4x \\ trA_B = 4\cdot\delta(0)\int\lambda(x)d^4x = 4\delta(0)\int\lambda(x)d^4x \end{cases}$$

最后得（加减号按 ***boson*** 或 ***fermion*** 而定）

$$\begin{aligned} J[\lambda] &= \left(1-6\delta(0)\int\lambda(x)d^4x\right)\left(1-6\delta(0)\int\lambda(x)d^4x\right)\left(1+4\delta(0)\int\lambda(x)d^4x\right) \\ &= 1-8\delta(0)\int\lambda(x)d^4x \end{aligned}$$

若对$\lambda(x)$求泛函导数（这是常常需要的），有

$$\boxed{\frac{\delta J(\lambda)}{\delta\lambda(y)} = -8\delta(0)} \qquad \text{(3.26b)}$$