[高等量子理论专题系列讲座]

近代量子理论中的几个数学专题

提 纲

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[第 3 讲]

泛函(路径)积分的数学分析

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一，泛函积分的分部积分运算

设算符$\hat{K}$是厄米的，且其中质量有一小的负虚部，于是它将无奇性，可以有逆算符。$\hat{K}$的定义是，若为积分形式，则为

$$\hat{K}: \quad \varphi \to \psi = \hat{K}\varphi$$
$$\psi(x) = \hat{K}\varphi(x) = \int dy\, K(x,y)\varphi(y) \tag{3.1}$$

这里$K(x,y)$是$\hat{K}$在函数空间的表示。而$K^{-1}(x,y)$则是逆算符$\hat{K}^{-1}$在函数空间的表示，是其逆变换的积分核；如是微分形式，可设$K^{-1}(x,y)$为算符$\hat{K}(x)$的 ***Green*** 函数，它满足如下方程，

$$\hat{K}(x)K^{-1}(x,y) = \delta(x-y) \tag{3.2}$$

显然，如果$\hat{K}(x)$是某个场方程左边的微分算符，则$K^{-1}(x,y) = K^{-1}(x-y)$就是该场的 ***Feynman*** 传播子（至多差一常数相因子）。可以直接解出这个 ***Green*** 函数为

$$K^{-1}(x,y) = \frac{1}{\hat{K}(x)}\delta(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \frac{1}{\hat{K}\left(\partial_\mu^{(x)} \to ip_\mu\right)} \tag{3.3}$$

1，分部积分定理

[定理] 若泛函积分$I = \int D\varphi\, F[\varphi]\, G[\varphi]$存在，并满足

$$\int D\varphi \frac{\delta\left(F[\varphi]G[\varphi]\right)}{\delta\varphi(x)} = 0$$

则有分部积分公式如下

$$\boxed{\int D\varphi \frac{\delta F[\varphi]}{\delta\varphi(x)} G[\varphi] = -\int D\varphi\, F[\varphi] \frac{\delta G[\varphi]}{\delta\varphi(x)}} \tag{3.4}$$

证明：

$$\text{左边} = \int D\varphi \left\{ \frac{\delta\left(F[\varphi]G[\varphi]\right)}{\delta\varphi(x)} - F[\varphi]\frac{\delta G[\varphi]}{\delta\varphi(x)} \right\} = -\int D\varphi\, F[\varphi]\frac{\delta G[\varphi]}{\delta\varphi(x)}$$

2，分部积分计算举例。设有如下含外源的泛函积分

$$I[\eta]=\int D\varphi\exp\left\{i\int dx'\varphi(x')\eta(x')\right\}\frac{\delta}{\delta\varphi(x)}\exp\left\{\frac{i}{2}\int dx'dy'\varphi(x')K(x'-y')\varphi(y')\right\}$$

可证有如下分部积分公式

$$\boxed{\begin{aligned}&\int D\varphi\exp\left\{i\int dx'\varphi(x')\eta(x')\right\}\left(\frac{\delta}{\delta\varphi(x)}\exp\left\{\frac{i}{2}\int dx'dy'\varphi(x')K(x'-y')\varphi(y')\right\}\right)\\&=-\int D\varphi\exp\left\{\frac{i}{2}\int dx'dy'\varphi(x')K(x'-y')\varphi(y')\right\}\left(\frac{\delta}{\delta\varphi(x)}\exp\left\{i\int dx'\varphi(x')\eta(x')\right\}\right)\end{aligned}}$$

（3.5）

证明：现往证

$$\int D\varphi\frac{\delta}{\delta\varphi(x)}\exp\left\{\frac{i}{2}\int dx'dy'\varphi(x')K(x'-y')\varphi(y')+i\int dx'\varphi(x')\eta(x')\right\}=0$$

引入记号 $Z[\eta]$ 并在下面 (二,2) 节给出此表达式的证明：

$$\boxed{\begin{aligned}Z[\eta]&\equiv\int D\varphi\exp\left\{\frac{i}{2}\int dx'dy'\varphi(x')K(x'-y')\varphi(y')+i\int dx'\varphi(x')\eta(x')\right\}\\&=\frac{1}{\sqrt{Det\,\hat{K}}}\exp\left\{\frac{-i}{2}\int dzdz'\eta(z')K^{-1}(z'-z)\eta(z)\right\}\end{aligned}}\quad(3.6)$$

于是

$$\int D\varphi\frac{\delta}{\delta\varphi(x)}\exp\left\{\frac{i}{2}\int dx'dy'\varphi(x')K(x'-y')\varphi(y')+i\int dx'\varphi(x')\eta(x')\right\}$$

$$=\int D\varphi\left\{i\int dy'K(x-y')\varphi(y')+i\eta(x)\right\}\exp\left\{\frac{i}{2}\int dx'dy'\varphi K\varphi+i\int dx'\varphi\eta\right\}$$

$$=i\int dy'K(x-y')\int D\varphi\,\varphi(y')\exp\left\{\frac{i}{2}\int dx'dy'\varphi(x')K(x'-y')\varphi(y')+i\int dx'\varphi\eta\right\}$$
$$+i\eta(x)Z[\eta]$$

$$=i\int dy'K(x-y')\frac{\delta Z[\eta]}{i\delta\eta(y')}+i\eta(x)Z[\eta]$$

$$=\int dy'K(x-y')\frac{\delta}{\delta\eta(y')}\frac{1}{\sqrt{Det\hat{K}}}\exp\left\{\frac{-i}{2}\int dzdz'\eta(z')K^{-1}(z'-z)\eta(z)\right\}+i\eta(x)Z[\eta]$$

$$= \frac{-i}{\sqrt{Det\hat{K}}} \int dy' dz K(x-y') K^{-1}(y'-z) \eta(z) \exp\left\{ \frac{-i}{2} \int dz dz' \eta K^{-1} \eta \right\} + i\eta(x) Z[\eta]$$

$$= -i \int dy' dz K(x-y') K^{-1}(y'-z) \eta(z) Z[\eta] + i\eta(x) Z[\eta]$$

$$= -i\eta(x) Z[\eta] + i\eta(x) Z[\eta] = 0$$

证毕。

注意，此处分部积分公式（3.5）右边将为

$$-\int D\varphi \exp\left\{ \frac{i}{2} \int dx' dy' \varphi(x') K(x'-y') \varphi(y') \right\} \left(\frac{\delta}{\delta\varphi(x)} \exp\left\{ i \int dx' \varphi(x') \eta(x') \right\} \right)$$

$$= -i\eta(x) \int D\varphi \exp\left\{ \frac{i}{2} \int dx' dy' \varphi(x') K(x'-y') \varphi(y') + i \int dx' \varphi(x') \eta(x') \right\}$$

$$= \frac{-i\eta(x)}{\sqrt{Det\hat{K}}} \exp\left\{ \frac{-i}{2} \int dx' dy' \eta(x') K^{-1}(x'-y') \eta(y') \right\} \qquad (3.7)$$

二，Gauss 型泛函积分计算举例

1，作为预备，下面列举几个 *Gauss* 型多重积分等式。

《多实变数情况》（A 为正定对称矩阵）：

$$\begin{cases} \int_{-\infty}^{\infty} \exp\left(-A_{ij} x_i x_j\right) \prod_{i=1}^{n} dx_i = \sqrt{\frac{\pi^n}{\det A}} \\ \int_{-\infty}^{\infty} \exp\left(-A_{ij} x_i x_j + \alpha_i x_i\right) \prod_{i=1}^{n} dx_i = \sqrt{\frac{\pi^n}{\det A}} \exp\left(\frac{1}{4} \tilde{\alpha} A^{-1} \alpha \right) \end{cases}$$

《多复变数情况》（A 的实部为正定对称矩阵）：

$$\begin{cases} \int_{-\infty}^{\infty} \exp\left(-Z^{+} A Z + \beta^{+} Z + Z^{+} \beta\right) \prod_{i=1}^{n} \frac{d^2 z_i}{\pi} = \frac{1}{\det A} \exp\left(\beta^{+} A^{-1} \beta\right), \quad d^2 z_i = dx_i dy_i \\ \int_{-\infty}^{\infty} \exp\left(-Z^{+} A Z - \frac{1}{2} \tilde{Z} B Z - \frac{1}{2} Z^{+} C Z^{*} + \alpha^{+} Z + Z^{+} \beta \right) \prod_{i=1}^{n} \frac{d^2 z_i}{\pi} = \\ \qquad = \left[\det \begin{pmatrix} B & \tilde{A} \\ A & C \end{pmatrix} \right] \exp\left\{ \frac{1}{2} \left(\alpha^{+}, \tilde{\beta}\right) \begin{pmatrix} B & \tilde{A} \\ A & C \end{pmatrix}^{-1} \begin{pmatrix} \alpha^{*} \\ \beta \end{pmatrix} \right\}, \quad \tilde{B} = B, \tilde{C} = C \end{cases}$$

《多 *Grassmann* 数情况》：

$$\begin{cases} \int \exp\left(-\frac{1}{2}\tilde{\tau}A\tau\right)\prod_{i=1}^{n} d\tau_i = \sqrt{\det A}\,, \quad \tilde{A} = -A\,, \quad \prod_{i=1}^{n} d\tau_i = d\tau_1 \cdots d\tau_n \\ \int \exp\left(-\frac{1}{2}\tilde{\tau}A\tau + \tilde{\alpha}\tau\right)\prod_{i=1}^{n} d\tau_i = \sqrt{\det A}\exp\left(-\frac{1}{2}\tilde{\alpha}A^{-1}\alpha\right) \\ \int \exp\left(-\tau^{+}A\tau + \sigma^{+}\tau + \tau^{+}\sigma\right)\prod_{i=1}^{n} d\tau_i d\tau_i^{*} = \det A\exp\left(\sigma^{+}A^{-1}\sigma\right) \end{cases}$$

$$\int \exp\left(-\tau^{+}A\tau - \frac{1}{2}\tilde{\tau}B\tau - \frac{1}{2}\tau^{+}C\tilde{\tau}^{+} + \tilde{\alpha}\tau + \tau^{+}\beta\right)\prod_{i=1}^{n} d\tau_i d\tau_i^{*} =$$

$$= \left[\det\begin{pmatrix} B & -\tilde{A} \\ A & C \end{pmatrix}\right]^{1/2} \exp\left\{-\frac{1}{2}\left(\tilde{\alpha},-\tilde{\beta}\right)\begin{pmatrix} B & -\tilde{A} \\ A & C \end{pmatrix}^{-1}\begin{pmatrix} \alpha \\ -\beta \end{pmatrix}\right\}$$

2，实标场 *Gauss* 型二次齐次型泛函积分（$x = (\vec{x}, x_4 = ict)$）

$$\int D\varphi(x)\exp\left\{-\frac{1}{2}\int dxdy\,\varphi(x)\hat{K}(x,y)\varphi(y)\right\} = \frac{1}{\sqrt{Det\hat{K}}} \qquad \text{(3.8a)}$$

这里测度 $D\varphi$ 的选取保证积分等式成立。

证明： 从有限维对角看，这里指数上非对角形式的双重求和 $\int dxdy\,\varphi(x)K(x,y)\varphi(y)$ 与对角形式的单重求和 $\int dx\,\varphi(x)K(x,x)\varphi(x)$ 只相差一个对 $\varphi(x)$ 的矩阵变换，或者说对中间矩阵 $\hat{K}$ 的相似变换。两者积分至多相差一个矩阵变换 *Jacobi*，将来和分母上同样因子相消。这由（3.8）式结果行列式 $det\hat{K} = det\left(C^{-1}\hat{K}C\right)$ 也可理解。于是只需要证明对角形式即可，

$$\int D\varphi(x)\exp\left\{-\frac{1}{2}\int dx\,\varphi(x)\hat{K}(x,x)\varphi(x)\right\} = \frac{1}{\sqrt{Det\hat{K}}}$$

将它看成是有限维对角情况向连续无限维的推广。按直接计算办法，

$$左 = \lim_{\substack{\Delta x\to 0 \\ n\to\infty}}\int\prod_{i=1}^{n}\frac{d\varphi(x_i)}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\Delta x\cdot\varphi(x_i)K(x_i,x_i)\cdot\varphi(x_i)\right\}$$

$$= \lim_{\substack{\Delta x\to 0 \\ n\to\infty}}\int\prod_{i=1}^{n}\frac{d\varphi_i}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\varphi_i\left(K_{ii}\Delta x\right)\cdot\varphi_i\right\} = \frac{1}{\sqrt{Det\hat{K}}}$$

最后一步是，针对取极限前对角化有限重积分到取极限后无限重积

分（即路径积分或泛函积分）过渡，采用有限维对角矩阵行列式向无限维算符行列式的转换。此时Δx自动含在算符行列式求迹的积分运算之中。此式也常写作

$$\boxed{\int D\varphi(x)\exp\left\{-\frac{1}{2}\int dx\,\varphi(x)\hat{K}\varphi(x)\right\}=\frac{1}{\sqrt{Det\hat{K}}}} \qquad (3.8b)$$

总之，在x取值分立的有限维情况下，普通函数$K(x)$对其后函数$\varphi(x)$的作用相当于对角矩阵；算符$\hat{K}$对其后函数$\varphi(x)$的作用相当于非对角矩阵；而在函数空间中，算符$\hat{K}$的作用可视作$\hat{K}\varphi(x)=\int dyK(x,y)\varphi(y)$。

3，实标场 *Gauss* 型非齐次二次型泛函积分

$$\boxed{\begin{aligned}&\int D\varphi(x)\exp\left\{\int dx\exp\left(-\frac{1}{2}\varphi(x)\hat{K}\varphi(x)+J(x)\varphi(x)\right)\right\}\\&=\frac{1}{\sqrt{Det\hat{K}}}\exp\left\{\frac{1}{2}\int d(xy)J(x)K^{-1}(x,y)J(y)\right\}\end{aligned}} \qquad (3.9a)$$

测度$D\varphi$的选取应保证此积分在$J=0$时回归到等式（3.8b）。

证明：这也是有限模情况向连续模情况的推广。作函数平移变换（相应的泛函 *Jacobi* 等于 1，说明见结尾处）：

$$\varphi(x)\to\Phi(x)=\varphi(x)-\int d^4x'K^{-1}(x,x')J(x')$$

于是有

$$\begin{aligned}&\frac{-1}{2}\int d^4x\,\Phi(x)\hat{K}\Phi(x)=\\&=\frac{-1}{2}\int d^4x\left[\varphi(x)-\int d^4x'K^{-1}(x,x')J(x')\right]\hat{K}\left[\varphi(x)-\int d^4x''K^{-1}(x,x'')J(x'')\right]\\&=\frac{-1}{2}\int d^4x\varphi(x)\hat{K}\varphi(x)+\frac{1}{2}\int d^4(xx')K^{-1}(x,x')J(x')\hat{K}\varphi(x)\\&\quad+\frac{1}{2}\int d^4(xx'')\varphi(x)\hat{K}K^{-1}(x,x'')J(x'')\\&\quad-\frac{1}{2}\int d^4(xx'x'')K^{-1}(x,x')J(x')\hat{K}K^{-1}(x,x'')J(x'')\end{aligned}$$

$$= \frac{-1}{2}\int d^4x\,\varphi(x)\hat{K}\varphi(x) + \frac{1}{2}\int d^4(xx')K^{-1}(x,x')J(x')\hat{K}\varphi(x)$$
$$+\frac{1}{2}\int d^4x\,\varphi(x)J(x) - \frac{1}{2}\int d^4(xx')K^{-1}(x,x')J(x')J(x)$$

如果$\hat{K}(x)$中含$\partial_\mu^{(x)}$二阶偏导数，则第二项经分部积分两次，从而可知第二、第三两项相等。将第四项移至等式左边，即得下面等式：

$$\frac{-1}{2}\int d^4x\,\varphi\hat{K}\varphi + \int d^4x\varphi J = \frac{-1}{2}\int d^4x\Phi(x)\hat{K}\Phi(x) + \frac{1}{2}\int d^4(xy)J(x)K^{-1}(x,y)J(y)$$

将此式右边替代（**3.9a**）式左边的指数，并利用（**3.8b**）式，即得

$$\int D\Phi\exp\left\{\frac{-1}{2}\int d^4x\,\Phi(x)\hat{K}\Phi(x) + \frac{1}{2}\int d^4(xy)J(x)K^{-1}(x,y)J(y)\right\}$$
$$= \frac{1}{\sqrt{Det\hat{K}}}\exp\left\{\frac{1}{2}\int d^4(xy)J(x)K^{-1}(x,y)J(y)\right\}$$ 证毕。

最后，说明此变换的 ***Jacobi*** 为 **1**。即便对多分量的实标场，由于

$$\frac{\delta\varphi_i(x)}{\delta\Phi_j(y)} = \delta_{ij}\delta(x-y) = \delta_{ij}\langle x|\hat{I}|y\rangle$$

于是$J\left(\frac{\delta\varphi_i}{\delta\Phi_j}\right) = \det\hat{I} = \exp\{tr\ln\hat{I}\} = 1$。（**3.9a**）式也常写作

$$\int D\varphi(x)\exp\left\{\frac{i}{2}\int d(xy)\varphi(x)K(x,y)\varphi(y) + i\int dx\,\varphi(x)J(x)\right\}$$
$$= \frac{1}{\sqrt{Det\hat{K}}}\exp\left\{\frac{-i}{2}\int d(xy)J(x)K^{-1}(x,y)J(y)\right\} \tag{3.9b}$$

此式的具体算例是：有质量的矢量玻色子场。即令

$$\hat{K}_{\mu\nu}(x) = \left[\delta_{\mu\nu}\left(\partial_\mu^{(x)2} - M^2\right) - \partial_\mu^{(x)}\partial_\nu^{(x)}\right],\quad M^2 \to M^2(1-i\varepsilon)$$

$$\int\prod_\mu DB_\mu\exp\left\{i\int d^4x\left[\frac{1}{2}B_\mu(x)\hat{K}_{\mu\nu}(x)B_\nu(x) + J_\mu(x)B_\mu(x)\right]\right\}$$
$$= \frac{1}{\sqrt{\left|Det\hat{K}_{\mu\nu}\right|}}\exp\left\{-\frac{1}{2}\int d^4(xy)J_\mu(x)D_F(x,y)_{\mu\nu}J_\nu(y)\right\} \tag{3.9c}$$

（3.9c）式的测度为 $DB_\mu = D\left(\frac{B_\mu}{\sqrt{2\pi i}}\right)$。同时有

$$\hat{K}_{\mu\nu} D_F(x,y)_{\nu\lambda} = i\delta_{\mu\lambda}\delta(x-y),\quad D_F(x,y)_{\nu\lambda} = -i\int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} \frac{\left(\delta_{\nu\lambda} + k_\nu k_\lambda / k^2\right)}{\sqrt{Det\hat{K}_{\mu\nu}}}$$

$$\hat{K}_{\mu\nu}\left(\hat{K}\right)^{-1}_{\nu\lambda}(x,y) = \delta_{\mu\lambda}\delta(x-y)$$

4，复标场情况

$$\int D\varphi D\varphi^* \exp\left\{-\int dxdy\left[\varphi^+(x)A(x,y)\varphi(y) + \frac{1}{2}\tilde{\varphi}(x)B(x,y)\varphi(y) + \frac{1}{2}\varphi^+(x)C(x,y)\tilde{\varphi}^+(y)\right] + \int dx\left[J^+(x)\varphi(x) + \varphi^+(x)J(x)\right]\right\}$$

$$= \left[\det\begin{pmatrix}\hat{B} & \hat{\tilde{A}} \\ \hat{A} & \hat{C}\end{pmatrix}\right]^{-1/2} \exp\left\{\frac{1}{2}\int dxdy\begin{pmatrix}J^+(x) & \tilde{J}(x)\end{pmatrix}\begin{pmatrix}B(x,y) & \tilde{A}(x,y) \\ A(x,y) & C(x,y)\end{pmatrix}^{-1}\begin{pmatrix}\tilde{J}^+(y) \\ J(y)\end{pmatrix}\right\}$$

（3.10a）

这里测度为 $D\varphi D\varphi^* = \prod_x \frac{(\Delta x)^4}{2\pi} d\varphi d\varphi^*$， $\tilde{A}(x,y) = A(y,x)$。

证明： （3.10a）式是有限模情况向连续模情况的推广。左边为

$$\int D\varphi D\varphi^* \exp\left\{-\frac{1}{2}\int dxdy\begin{pmatrix}\tilde{\varphi}(x) & \varphi^+(x)\end{pmatrix}\begin{pmatrix}B(x,y) & A(y,x) \\ A(x,y) & C(x,y)\end{pmatrix}\begin{pmatrix}\varphi(y) \\ \tilde{\varphi}^+(y)\end{pmatrix} + \int dx\begin{pmatrix}J^+(x) & \tilde{J}(x)\end{pmatrix}\begin{pmatrix}\varphi(y) \\ \tilde{\varphi}^+(y)\end{pmatrix}\right\}$$

利用（3.9b）式，得

$$\text{左} = \left(\left|\det\begin{pmatrix}B & \tilde{A} \\ A & C\end{pmatrix}\right|\right)^{-1/2} \exp\left\{\frac{1}{2}\int dxdy\begin{pmatrix}J^+(x) & \tilde{J}(x)\end{pmatrix}\begin{pmatrix}B & \tilde{A} \\ A & C\end{pmatrix}^{-1}(x,y)\begin{pmatrix}\tilde{J}^+(y) \\ J(y)\end{pmatrix}\right\}$$

证毕。

（3.10a）式具体算例是复标量场：$\hat{A}(x) = -i\left(\partial_\mu^{(x)2} - m^2\right)$， $m^2 \to m^2 - i\varepsilon$，

$$\boxed{\begin{aligned}&\int D\varphi D\varphi^* \exp\left\{i\int d^4x\left[\varphi^*\partial_\mu^2\varphi - m^2\varphi^*\varphi + J^*\varphi + J\varphi^*\right]\right\} = \\ &= \left|Det\left(\partial_\mu^2 - m^2\right)\right|^{-1} \exp\left\{-\int d^4(xy) J^*(x)\Delta_F(x,y)J(y)\right\}\end{aligned}} \qquad (3.10b)$$

此处测度为 $D\varphi D\varphi^* = \frac{D\varphi}{\sqrt{\pi i}}\frac{D\varphi^*}{\sqrt{\pi i}}$，并且有

$$\hat{A}(x)A^{-1}(x,y) = \delta(x-y) \rightarrow A^{-1}(x,y) = \Delta_F(x,y)$$

5，*Gauss* 型二次齐次 *Grassmann* 数泛函积分

$$\boxed{\int D\psi D\bar{\psi} \exp\left\{-\int dx\,\bar{\psi}K\psi\right\} = DetK} \qquad (3.11)$$

证明：这是向连续无穷自由度的推广，可由相应式证明过程得知。

6，*Gauss* 型二次非齐次 *Grassmann* 数泛函积分

设 $A(x)$ 为普通函数，可证有

$$\boxed{\begin{aligned}&\int D\psi D\bar{\psi} \exp\left\{\int dx\left[-\bar{\psi}K\psi + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)\right]\right\} = \\ &= |DetK| \exp\left\{\int dx\,\bar{\eta}(x)K^{-1}(x,x)\eta(x)\right\}\end{aligned}} \qquad (3.12a)$$

或

$$\boxed{\begin{aligned}&\int D\psi D\bar{\psi} \exp\left\{i\int dx\left[-\bar{\psi}K\psi + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)\right]\right\} = \\ &= |DetK| \exp\left\{-i\int dx\,\bar{\eta}(x)K^{-1}(x,x)\eta(x)\right\}\end{aligned}} \qquad (3.12b)$$

证明：在函数空间中（取定一组正交完备基，将 ψ 和 $\bar{\psi}$ 展开，知这时是对角矩阵情况），普通函数 $K(x)$ 对应的矩阵是对角的，于是直接使用有限模情况公式即可。

对于 $K(x)$ 是算符的情况，有

$$\boxed{\begin{aligned}&\int D\psi D\bar{\psi} \exp\left\{i\int dx\left[-\bar{\psi}\hat{K}\psi + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)\right]\right\} = \\ &= \left|Det\hat{K}\right| \exp\left\{-i\int dxdy\,\bar{\eta}(x)K^{-1}(x,y)\eta(y)\right\}\end{aligned}} \qquad (3.12c)$$

此式具体算例是旋量场：$\hat{K}(x) = -\left(\gamma_\mu\partial_\mu^{(x)} + \kappa\right)$，$\kappa = m - i\varepsilon$ 。$\hat{K}^{-1} = \hat{K}^{-1}(x,y)$ 的定义是：用它右乘 $\hat{K}(x)$ 后等于 $\delta(x-y)$，即

$$\hat{K}(x)\hat{K}^{-1}(x,y)=-\left(\gamma_\mu\partial_\mu^{(x)}+\kappa\right)K^{-1}(x,y)=\delta(x-y)$$

$$\therefore\quad K^{-1}(x,y)=\frac{1}{\hat{K}(x)}\delta(x-y)=\int\frac{d^4p}{(2\pi)^4}e^{ip(x-y)}\frac{-1}{i\not{p}+m(1-i\varepsilon)}=-iS_F(x,y)$$

证明：这时在函数空间中，$\hat{K}$ 对其后$\psi(x)$的作用相当于非对角矩阵。此时$\hat{K}^{-1}$定义见前面叙述。证明也直接用有限模公式，

左边=

$$=\int D\psi D\bar{\psi}\exp\left\{\int d^4x\left[-\bar{\psi}(x)\left(-i\hat{K}(x)\right)\psi(x)+\left(i\bar{\eta}(x)\right)\psi(x)+\bar{\psi}(x)\left(i\eta(x)\right)\right]\right\}$$

$$=\left|Det\left(-i\hat{K}(x)\right)\right|\exp\left\{\int d^4(xy)\left(i\bar{\eta}(x)\right)\left(-i\hat{K}(x)\right)^{-1}\left(i\eta(y)\right)\right\}$$

$$=\left|Det\hat{K}(x)\right|\exp\left\{-i\int d^4(xy)\bar{\eta}(x)\left(\hat{K}(x)\right)^{-1}\eta(y)\right\}$$

三，泛函$\delta-$函数

1，泛函$\delta-$函数的两种表达式

i，第一种表达式

$$\boxed{\delta[\Omega]=\prod_x\delta(\Omega(x))=\int[D\xi(x)]\exp\left\{i\int d^4x'\Omega(x')\xi(x')\right\}}\qquad(3.13a)$$

或

$$\boxed{\delta[\varphi-\psi]=\prod_x\delta(\varphi(x)-\psi(x))=\int[D\xi(x)]\exp\left\{i\int d^4x'(\varphi(x')-\psi(x'))\xi(x')\right\}}$$

（3.13b）

并有

$$\boxed{\int D\varphi F[\varphi]\delta[\varphi-\psi]=F[\psi]}\qquad(3.14)$$

对 *Gauss* 型泛函$F[\varphi]$情况，（3.14）式可以直接证明如下：

证明： 设泛函$F[\varphi]=\exp\left\{\frac{i}{2}\int d^4(xy)\varphi(x)K(x-y)\varphi(y)\right\}$，将此泛函乘以泛函$\delta-$函数并作泛函积分，为

左边=

$$= \int D\varphi \exp\left\{\frac{i}{2}\int d^4(xy)\varphi(x)K(x-y)\varphi(y)\right\}\int D\xi \exp\left\{i\int d^4x\left[\varphi(x)-\psi(x)\right]\xi(x)\right\}$$

$$= \int D\xi \exp\left\{-i\int d^4x\,\psi(x)\xi(x)\right\}\int D\varphi \exp\left\{\frac{i}{2}\int d^4(xy)\varphi(x)K(x-y)\varphi(y)+\right.$$
$$\left. + i\int d^4x\,\varphi(x)\xi(x)\right\}$$

$$= \int D\xi \exp\left\{-i\int d^4x\,\psi(x)\xi(x)\right\}\frac{1}{\sqrt{\left|Det\hat{K}\right|}}\exp\left\{-\frac{i}{2}\int d^4(xy)\xi(x)K^{-1}(x-y)\xi(y)\right\}$$

$$= \frac{1}{\sqrt{\left|Det\hat{K}\right|}}\int D\xi \exp\left\{\frac{i}{2}\int d^4(xy)\xi(x)\left[-K^{-1}(x-y)\right]\xi(y)+i\int d^4x\left[-\psi(x)\right]\xi(x)\right\}$$

$$= \frac{1}{\sqrt{\left|Det\hat{K}\right|}}\frac{1}{\sqrt{\left|Det\left(-\hat{K}^{-1}\right)\right|}}\exp\left\{\frac{i}{2}\int d^4(xy)\psi(x)K(x-y)\psi(y)\right\}$$

$$= \exp\left\{\frac{i}{2}\int d^4(xy)\psi(x)K(x-y)\psi(y)\right\} = F[\psi]$$ 证毕。

由此处泛函$\delta-$函数定义（3.13b）式可知，泛函$\delta-$函数为偶函数。

ii, **第二种表达式**

$$\delta\left[F_\alpha[\varphi]\right] \equiv \prod_{x,\alpha}\delta\left(F_\alpha[\varphi]\right) = \lim_{\varepsilon\to 0}\exp\left\{-\frac{i}{2\varepsilon}\sum_\alpha\int d^4x\left(F_\alpha[\varphi]\right)^2\right\} \qquad (3.15)$$

此公式将 *Gauss* 型泛函$\delta-$函数表示推广成为 *Fresnel* 型的表示。若分立指标α为指定的，则右边指数没有对α求和。

证明： 首先将普通$\delta-$函数的 *Gauss* 型表示推广到 *Fresnel* 型表示；

$$\delta(x) = \lim_{b\to 0}\frac{1}{\sqrt{\pi b}}e^{-x^2/b} \to \delta(x) = \lim_{b\to 0}\frac{e^{i\pi/4}}{\sqrt{\pi b}}e^{-ix^2/b}$$

于是有

$$\delta\left[F_\alpha[\varphi]\right] \equiv \prod_{x,\alpha}\delta\left(F_\alpha[\varphi]\right) \propto \lim_{b\to 0}\exp\left\{-\frac{i}{b}\sum_{x,\alpha}\left(F_\alpha[\varphi]\right)^2\right\}$$

$$= \lim_{\substack{b\to 0\\ \Delta x\to 0}} \exp\left\{ -\frac{i}{b(\Delta x)^4}\sum_{\alpha}\int d^4x\left(F_{\alpha}[\varphi]\right)^2 \right\}$$

$$= \lim_{\varepsilon\to 0} \exp\left\{ \frac{-i}{2\varepsilon}\sum_{\alpha}\int d^4x\left(F_{\alpha}[\varphi]\right)^2 \right\}$$

这里在“$\propto$”号的一步中略去了δ－函数中与函数$\varphi(x)$无关的常数因子，因为它与生成泛函分母归一化系数中的对应因子相消。

2，泛函δ－函数的自变量变换

i, 泛函δ－函数为偶函数。即总有

$$\boxed{\delta[\varphi-\psi]=\delta[\psi-\varphi]} \qquad (3.16)$$

由第一种表达式（3.13b）可知，这相当于对泛函积分变数做变换$\xi(x)\to-\xi(x)$时，积分数值不变。这只需注意，做此变换时泛函积分上下限也变换，所以有$\int\left[D\xi(x)\right]=\int\left[D(-\xi(x))\right]$。

ii, 泛函δ－函数的自变量变换

$$\boxed{\delta\left[\varphi-\hat{M}\psi\right]=\left|Det\hat{M}\right|^{-1}\delta\left[\psi-\hat{M}^{-1}\varphi\right]} \qquad (3.17)$$

这里$\delta\left[\varphi-\hat{M}\psi\right]=\prod_{x}\delta\left(\varphi(x)-\left(\hat{M}\psi\right)(x)\right)$。

证明： 有映射$\hat{M}:\psi(x)\to\varphi(x)$（即$\hat{M}\psi=\varphi$），$F[\varphi]=F\left[\varphi[\psi]\right]\equiv G[\psi]$。

设$\tilde{\varphi}$和$\tilde{\psi}$为满足$\hat{M}\tilde{\psi}=\tilde{\varphi}$的任意一组函数对，按泛函$\delta$－函数定义有

$$\begin{cases}\int D\varphi\cdot\delta\left[\varphi-\hat{M}\tilde{\psi}\right]F[\varphi]=F\left[\hat{M}\tilde{\psi}\right]\\ \int D\psi\cdot\delta\left[\psi-\hat{M}^{-1}\tilde{\varphi}\right]G[\psi]=G\left[\hat{M}^{-1}\tilde{\varphi}\right]\end{cases}$$

于是第一式的右边和左边分别等于

$$\begin{cases}F\left[\hat{M}\tilde{\psi}\right]=F[\tilde{\varphi}]=G[\tilde{\psi}]=G\left[\hat{M}^{-1}\tilde{\varphi}\right]\\ \int D\psi\left|Det\frac{\delta\varphi}{\delta\psi}\right|\delta\left[\varphi-\hat{M}\tilde{\psi}\right]F\left[\varphi[\psi]\right]=\int D\psi\left|Det\hat{M}\right|\delta\left[\varphi-\hat{M}\tilde{\psi}\right]G[\psi]\end{cases}$$

与第二式相比较，并略去“□”，即得（3.17）式。　　　　证毕。

于是，$\delta-$函数的前面定义又可以改写为：

$$\delta\left[\varphi-\hat{M}\psi\right]=\prod_x \delta\left(\varphi(x)-\int dyM(x-y)\psi(y)\right)=\left|Det\hat{M}\right|^{-1}\delta\left[\psi-\hat{M}^{-1}\varphi\right]$$

$$=\left|Det\hat{M}\right|^{-1}\prod_x \delta\left(\psi(x)-\int dyM^{-1}(x-y)\varphi(y)\right)$$

$$=\left|Det\hat{M}\right|^{-1}\int D\xi \exp\left\{i\int d^4x\left[\int M^{-1}(x-y)\varphi(y)dy-\psi(x)\right]\xi(x)\right\}$$

（3.18）

这些记法逻辑上是自洽的。因为对任意泛函$F[\varphi]$，按$\delta-$函数定义：

$$\int D\varphi\, F[\varphi]\delta\left[\varphi-\hat{M}\psi\right]=F\left[\hat{M}\psi\right]$$

另一方面，按泛函积分变换和$\delta-$函数变换的定义，上式左边为：

$$\int D\psi\left|Det\frac{\delta\varphi}{\delta\psi}\right|F\left[\varphi[\psi]\right]\delta\left[\varphi-\hat{M}\psi\right]$$

$$=\int D\psi\left|Det\hat{M}\right|F\left[\varphi[\psi]\right]\delta\left[\varphi-\hat{M}\psi\right]$$

$$=\int D\psi\, F\left[\varphi[\psi]\right]\delta\left[\psi-\hat{M}^{-1}\varphi\right]=F\left[\varphi\left[\hat{M}^{-1}\varphi\right]\right]$$

$$=F\left[\hat{M}\hat{M}^{-1}\varphi\right]=F[\varphi]=F\left[\hat{M}\psi\right]$$

四，泛函 *Fourier* 变换

1，泛函 *Fourier* 变换定义

显然，下面 *Gauss* 型泛函积分公式，

$$Z[\eta]=\int D\varphi \exp\left\{\frac{i}{2}\int dxdy\,\varphi(x)K(x-y)\varphi(y)+i\int dx\,\varphi(x)\eta(x)\right\}$$
$$=\frac{1}{\sqrt{Det\hat{K}}}\exp\left\{\frac{-i}{2}\int dxdy\,\eta(x)K^{-1}(x-y)\eta(y)\right\}$$

（3.19）

是一个**泛函 *Fourier* 变换式**：中间的指数二次型的被积泛函$F[\varphi]$，

$$F[\varphi]=\exp\left\{\frac{i}{2}\int dxdy\,\varphi(x)K(x-y)\varphi(y)\right\}$$

是场量函数$\varphi(x)$的 *Gauss* 型泛函$F[\varphi]$，是此**泛函 *Fourier* 变换的原泛函**；右边最后积出的结果是外源函数$\eta(x)$的 *Gauss* 型泛函$Z[\eta]$，

是**泛函 *Fourier* 变换的像泛函**。就是说，这两个泛函之间的关系为：

“**泛函 $Z[\eta]$ 是泛函 $F[\varphi]$ 的 *Fourier* 变换**。”

更一般的说，称：

$$“\int D\varphi\,\varphi(x_1)\cdots\varphi(x_n)\exp\left\{\frac{i}{2}\int dxdy\,\varphi(x)K(x-y)\varphi(y)+i\int dx\,\varphi(x)\eta(x)\right\}$$

$$=\frac{1}{\sqrt{Det\hat{K}}}\frac{\delta}{i\delta\eta(x_1)}\cdots\frac{\delta}{i\delta\eta(x_n)}\exp\left\{\frac{-i}{2}\int dxdy\,\eta(x)K^{-1}(x-y)\eta(y)\right\} \quad (3.20a)$$

这个关于外源 $\eta(x)$ 的 *Gauss* 型泛函是下面关于场量函数 $\varphi(x)$ 的 *Gauss* 型泛函

$$F[\varphi]=\varphi(x_1)\cdots\varphi(x_n)\exp\left\{\frac{i}{2}\int dxdy\,\varphi(x)K(x-y)\varphi(y)\right\} \quad (3.20b)$$

的 *Fourier* 变换。”

一般说，**[泛函 *Fourier* 变换定义]**：

“**如果下面泛函积分 $Z[\eta]$ 存在，**

$$Z[\eta]=\int D\varphi(x)F[\varphi(x)]\exp\left\{i\int dx'\varphi(x')\eta(x')\right\}$$

称外源泛函 $Z[\eta]$ 为场量泛函 $F[\varphi]$ 的 *Fourier* 变换像泛函。”

2，例算。作为泛函 ***Fourier*** 变换和 $\delta-$ 函数的一个应用，求证如下结论：　在变数变换

$$\begin{aligned}\varphi(x)=\hat{M}\psi(x)&=\int M(x,y)\psi(y)dy\\&=c_0(x)+\psi(x)+\int N(x,y)\psi(y)dy\\&\equiv c_0(x)+\psi(x)+N[\psi]\end{aligned}$$

之下，有下述泛函积分的转换

$$\begin{aligned}&\int D\varphi \exp\left\{\frac{i}{2}\int \varphi(x)K(x-y)\varphi(y)dxdy + i\int dx\varphi(x)J(x)\right\} = \\ &= \int D\psi\, Det\left(1+\frac{\delta N}{\delta\psi}\right)\exp\left\{\frac{i}{2}\int (\hat{M}\psi)(x)K(x-y)(\hat{M}\psi)(y)dxdy \right. \\ &\quad \left. + i\int (\hat{M}\psi)(x)J(x)dx\right\}\end{aligned} \qquad (3.21)$$

证明： 变换前的被积函数是 *Gauss* 型，积分有定义。根据 ***Fourier*** 变换思想只需证明 ***Fourier*** 变换等式（**3.21**）两边的原函数相等即可。

显然，等式左边表达式（记作 $I[J]$）表明，其原函数 $\tilde{I}[\varphi]$ 是

$$\exp\left\{\frac{i}{2}\int \varphi(x)K(x-y)\varphi(y)dxdy\right\}$$

而等式右边表达式（记作 $I'[J]$）的原函数 $\tilde{I}'[\psi]$ 是

$$\tilde{I}'[\psi] = \int DJ\cdot I'[J]\cdot \exp\left\{-i\int dxJ(x)\psi(x)\right\}$$

$$\begin{aligned}&= \int D\psi \int DJ\;\; Det\left(1+\frac{\delta N}{\delta\psi}\right)\exp\left\{\frac{i}{2}\int (\hat{M}\psi)(x)K(x-y)(\hat{M}\psi)(y)dxdy\right\}\cdot \\ &\quad \cdot \exp\left\{i\int J(x)\left[(\hat{M}\psi)(x)-\psi(x)\right]dx\right\}\end{aligned}$$

$$\begin{aligned}&= \int D\psi\, Det\left(1+\frac{\delta N}{\delta\psi}\right)\exp\left\{\frac{i}{2}\int (\hat{M}\psi)(x)K(x-y)(\hat{M}\psi)(y)dxdy\right\}\cdot \\ &\quad \cdot\delta\left[(\hat{M}\psi)(x)-\psi(x)\right]\end{aligned}$$

$$\begin{aligned}&= \int D\psi\;\; Det\left(1+\frac{\delta N}{\delta\psi}\right)\exp\left\{\frac{i}{2}\int \psi(x)K(x-y)\psi(y)dxdy\right\}\cdot \\ &\quad \cdot\delta\left[(\hat{M}\psi)(x)-\psi(x)\right]\end{aligned}$$

$$= \int D\varphi \exp\left\{\frac{i}{2}\int \psi(x)K(x-y)\psi(y)dxdy\right\}\delta\left[\varphi(x)-\psi(x)\right]$$

$$= \exp\left\{\frac{i}{2}\int \kappa(x)K(x-y)\varphi(y)dxdy\right\}$$

五，泛函积分的变数变换与泛函 *Jacobi*

1，泛函积分的变数变换，泛函 *Jacobi*

设线性函数映射 $M: \varphi(x) \to \psi(x)$，老场量 $\varphi(x)$ 由新场量 $\psi(x)$ 按如下变换相互联系

$$\varphi_\alpha(x) = \int M_{\alpha\beta}(x,y)\psi_\beta(y)dy \equiv \left(\hat{M}\psi(x)\right)_\alpha \qquad \text{（3.22a）}$$

α,β 为内禀、外在全部附属空间的分量指标。在此泛函变数变换下，泛函积分变换为

$$\int F\left[\varphi_\alpha(x)\right]\prod_\alpha D\varphi_\alpha(x) = \int F\left[\psi_\beta(y)\right] J\left(\frac{\delta\varphi_\alpha}{\delta\psi_\beta}\right)\prod_\beta D\psi_\beta(y)$$

积分测度的相应变换为

$$D\varphi(x) = \prod_{\alpha,x}\frac{d\varphi_\alpha(x)}{\sqrt{C}} = \prod_{\alpha,x}\frac{\delta\varphi_\alpha(x)}{\delta\psi_\beta(x)}\cdot\prod_{\beta,x}\frac{d\psi_\beta(x)}{\sqrt{C}}$$

$$\therefore \qquad \boxed{D\varphi(x) = \left|Det\hat{M}\right| D\psi(x)} \qquad \text{（3.22b）}$$

$$\int F\left[\varphi_\alpha(x)\right]\prod_\alpha D\varphi_\alpha(x) = \int F\left[\psi_\beta(y)\right] J\left(\frac{\delta\varphi_\alpha}{\delta\psi_\beta}\right)\prod_\beta D\psi_\beta(y)$$

最后一步等号是因为

$$\left(\frac{\delta\varphi_\alpha(x)}{\delta\psi_\beta(y)}\right) = M(x,y) \to$$

$$\prod_x\left(\frac{\delta\varphi_\alpha(x)}{\delta\psi_\beta(x)}\right) = \exp\left\{\int dx \ln\left(\frac{\delta\varphi_\alpha(x)}{\delta\psi_\beta(x)}\right)\right\} = \exp\left\{\int dxdy\delta(x-y)\ln M(x,y)\right\}$$

$$= \exp\left\{tr\ln\hat{M}\right\} = Det\hat{M}$$

于是，泛函 ***Jacobi*** 为

$$\boxed{J\left(\frac{\delta\varphi_\alpha}{\delta\varphi'_\beta}\right) = \begin{cases} Det\hat{M} = \exp\left[tr\left(\ln\hat{M}\right)\right], & for\ boson \\ \left(Det\hat{M}\right)^{-1} = \exp\left[-tr\left(\ln\hat{M}\right)\right], & for\ fermion \end{cases}} \qquad \text{（3.23）}$$

总之，在此变数变换下，泛函积分的表达式改换为

$$\boxed{Z[\eta] = \int D\varphi F[\varphi] = \int D\psi \left|Det\hat{M}\right| F\left[\hat{M}\psi\right]} \qquad \text{（3.24）}$$

这里 $\varphi(x)$ 与 $\psi(x)$ 的关系由（**3.22a**）式决定。

2，无穷小泛函变分，泛函 *Jacobi*

如果场变数的变化是无穷小的泛函变分，即变换 $\hat{M}$ 接近于恒等变换 $\hat{M}=\hat{I}+\hat{A}$，$\hat{A}$ 为含有穷小常数的算符，如同经常做的那样。这时可对上面 *Jacobi* 作近似，从而写作更方便的形式：

$$\exp\left[\pm tr\left(\ln\hat{M}\right)\right]=\exp\left[\pm tr\left(\ln\left(\hat{I}+\hat{A}\right)\right)\right]\approx\exp\left[\pm tr\hat{A}\right]\approx\hat{I}\pm tr\hat{A}$$

最后可得，此无穷小变换的泛函 *Jacobi* 为

$$J\left(\frac{\delta\varphi_{\alpha}}{\delta\psi_{\beta}}\right)\approx\begin{cases}\hat{I}+tr\hat{A}, & for\ boson\\ \hat{I}-tr\hat{A}, & for\ fermion\end{cases}\qquad(3.25)$$

举例：泛函 $\int D\psi D\bar{\psi}\prod_{\mu}DB_{\mu}F\left(\psi,\bar{\psi},B_{\mu}\right)$ 经受如下场量变换

$$\begin{cases}\psi(x)\to\psi'(x)=\left(1+\frac{3}{2}\lambda(x)\right)\psi(x)\\ \bar{\psi}(x)\to\bar{\psi}'(x)=\left(1+\frac{3}{2}\lambda(x)\right)\bar{\psi}(x)\\ B_{\mu}(x)\to B'_{\mu}(x)=\left(1+\lambda(x)\right)B_{\mu}(x)\end{cases}\qquad(3.26a)$$

这时，泛函 *Jacobi* 为

$$J(\lambda)=J_{\psi}(\lambda)J_{\bar{\psi}}(\lambda)J_{B}(\lambda)$$

于是，比如对 ψ，有

$$\psi'(x)=\int\delta(x-y)\left(1+\frac{3}{2}\lambda(x)\right)\psi(y)d^{4}y$$

$$\therefore\qquad\langle x|\hat{M}|y\rangle=M(x,y)=\delta(x-y)\left(1+\frac{3}{2}\lambda(x)\right)$$

因此

$$\begin{cases}\langle x|\hat{M}_{\psi}-1|y\rangle=\frac{3}{2}\lambda(x)\delta(x-y), & for\ \psi-4\dim\\ \langle x|\hat{M}_{\bar{\psi}}-1|y\rangle=\frac{3}{2}\lambda(x)\delta(x-y), & for\ \bar{\psi}-4\dim\\ \langle x|\hat{M}_{B}-1|y\rangle=\lambda(x)\delta(x-y), & for\ B_{\mu}-4\dim\end{cases}$$

$$
\begin{cases}
tr\left(\hat{M}_{\psi}-1\right)=4\cdot\frac{3}{2}\delta(0)\int\lambda(x)d^4x=6\delta(0)\int\lambda(x)d^4x \\
tr\left(\hat{M}_{\bar{\psi}}-1\right)=4\cdot\frac{3}{2}\delta(0)\int\lambda(x)d^4x=6\delta(0)\int\lambda(x)d^4x \\
tr\left(\hat{M}_{B}-1\right)=4\cdot\delta(0)\int\lambda(x)d^4x=4\delta(0)\int\lambda(x)d^4x
\end{cases}
$$

最后得（加减号按 ***boson*** 或 ***fermion*** 而定）

$$
\begin{aligned}
J(\lambda)&=\left(1-6\delta(0)\int\lambda(x)d^4x\right)\left(1-6\delta(0)\int\lambda(x)d^4x\right)\left(1+4\delta(0)\int\lambda(x)d^4x\right) \\
&\approx 1-8\delta(0)\int\lambda(x)d^4x
\end{aligned}
$$

若对$\lambda(x)$求泛函导数（这是常常需要的），则有

$$
\boxed{\frac{\delta J(\omega)}{\delta\lambda(y)}=-8\delta(0)} \qquad \textbf{（3.26b）}
$$