[高等量子理论专题系列讲座]

近代量子理论中的几个数学专题

提 纲

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[第 3 讲]

泛函(路径)积分的数学分析

提 纲

一，泛函积分的分部积分运算

二，*Gauss* 型泛函积分计算举例

三，泛函 δ – 函数

四，泛函 *Fourier* 变换

五，泛函积分的变数变换与泛函 *Jacobi*

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一，泛函积分的分部积分运算

设算符 $\hat{K}$ 是厄米的，且其中质量有一小的负虚部，于是它将无奇性，可以有逆算符。$\hat{K}$ 的定义是，若为积分形式，则为

$$\hat{K}: \quad \varphi \to \psi = \hat{K}\varphi$$
$$\psi(x) = \hat{K}\varphi(x) = \int dy\, K(x,y)\varphi(y) \tag{3.1}$$

这里 $K(x,y)$ 是 $\hat{K}$ 在函数空间的表示。而 $K^{-1}(x,y)$ 则是逆算符 $\hat{K}^{-1}$ 在函数空间的表示，是其逆变换的积分核；如是微分形式，可设 $K^{-1}(x,y)$ 为算符 $\hat{K}(x)$ 的 *Green* 函数，它满足如下方程，

$$\hat{K}(x)K^{-1}(x,y) = \delta(x-y) \tag{3.2}$$

显然，如果 $\hat{K}(x)$ 是某个场方程左边的微分算符，则 $K^{-1}(x,y) = K^{-1}(x-y)$ 就是该场的 *Feynman* 传播子（至多差一常数相因子）。可以直接解出这个 *Green* 函数为

$$K^{-1}(x,y) = \frac{1}{\hat{K}(x)}\delta(x-y) = \int \frac{d^4p}{(2\pi)^4}e^{ip(x-y)}\frac{1}{\hat{K}\left(\partial_\mu^{(x)} \to ip_\mu\right)} \tag{3.3}$$

1，分部积分定理

[**定理**] 若泛函积分 $I = \int D\varphi\, F[\varphi]\,G[\varphi]$ **存在，并满足**

$$\int D\varphi \frac{\delta\left(F[\varphi]G[\varphi]\right)}{\delta\varphi(x)} = 0$$

则有分部积分公式如下

$$\boxed{\int D\varphi \frac{\delta F[\varphi]}{\delta\varphi(x)}G[\varphi] = -\int D\varphi\, F[\varphi]\frac{\delta G[\varphi]}{\delta\varphi(x)}} \tag{3.4}$$

证明：

$$左边 = \int D\varphi \left\{ \frac{\delta\left(F[\varphi]G[\varphi]\right)}{\delta\varphi(x)} - F[\varphi]\frac{\delta G[\varphi]}{\delta\varphi(x)} \right\} = -\int D\varphi\, F[\varphi]\frac{\delta G[\varphi]}{\delta\varphi(x)}$$

2，分部积分计算举例。设有如下含外源的泛函积分

$$I[\eta] = \int D\varphi \exp\left\{i\int dx'\varphi(x')\eta(x')\right\}\frac{\delta}{\delta\varphi(x)}\exp\left\{\frac{i}{2}\int dx'dy'\varphi(x')K(x'-y')\varphi(y')\right\}$$

可证有如下分部积分公式

$$\int D\varphi \exp\left\{i\int dx'\varphi(x')\eta(x')\right\}\left(\frac{\delta}{\delta\varphi(x)}\exp\left\{\frac{i}{2}\int dx'dy'\varphi(x')K(x'-y')\varphi(y')\right\}\right)$$

$$= -\int D\varphi \exp\left\{\frac{i}{2}\int dx'dy'\varphi(x')K(x'-y')\varphi(y')\right\}\left(\frac{\delta}{\delta\varphi(x)}\exp\left\{i\int dx'\varphi(x')\eta(x')\right\}\right)$$

（3.5）

证明：现往证

$$\int D\varphi \frac{\delta}{\delta\varphi(x)}\exp\left\{\frac{i}{2}\int dx'dy'\varphi(x')K(x'-y')\varphi(y') + i\int dx'\varphi(x')\eta(x')\right\} = 0$$

引入记号 $Z[\eta]$ 并在下面 **(二,2)** 节给出此表达式的证明：

$$Z[\eta] \equiv \int D\varphi \exp\left\{\frac{i}{2}\int dx'dy'\varphi(x')K(x'-y')\varphi(y') + i\int dx'\varphi(x')\eta(x')\right\}$$

$$= \frac{1}{\sqrt{Det\,\hat{K}}}\exp\left\{\frac{-i}{2}\int dzdz'\eta(z')K^{-1}(z'-z)\eta(z)\right\}$$

（3.6）

于是

$$\int D\varphi \frac{\delta}{\delta\varphi(x)}\exp\left\{\frac{i}{2}\int dx'dy'\varphi(x')K(x'-y')\varphi(y') + i\int dx'\varphi(x')\eta(x')\right\}$$

$$= \int D\varphi \left\{i\int dy'K(x-y')\varphi(y') + i\eta(x)\right\}\exp\left\{\frac{i}{2}\int dx'dy'\varphi K\varphi + i\int dx'\varphi\eta\right\}$$

$$= i\int dy'K(x-y')\int D\varphi\,\varphi(y')\exp\left\{\frac{i}{2}\int dx'dy'\varphi(x')K(x'-y')\varphi(y') + i\int dx'\varphi\eta\right\}$$

$$+ i\eta(x)Z[\eta]$$

$$= i\int dy'K(x-y')\frac{\delta Z[\eta]}{i\delta\eta(y')} + i\eta(x)Z[\eta]$$

$$= \int dy'K(x-y')\frac{\delta}{\delta\eta(y')}\frac{1}{\sqrt{Det\hat{K}}}\exp\left\{\frac{-i}{2}\int dzdz'\eta(z')K^{-1}(z'-z)\eta(z)\right\} + i\eta(x)Z[\eta]$$

$$= \frac{-i}{\sqrt{Det\hat{K}}} \int dy'dz K(x-y') K^{-1}(y'-z) \eta(z) \exp\left\{\frac{-i}{2}\int dzdz' \eta K^{-1}\eta\right\} + i\eta(x) Z[\eta]$$

$$= -i\int d'y\ d\ z\ (K\ -x)\ \vec{y}^1\ (\ K\ '\ -)\eta(\ )z\ [\ \eta]z\ \mathbb{Z}(\ )\ [i\ ]\eta$$

$$= -i\eta(x) Z[\eta] + i\eta(x) Z[\eta] = 0 \qquad\qquad 证毕。$$

注意，此处分部积分公式（**3.5**）右边将为

$$-\int D\varphi \exp\left\{\frac{i}{2}\int dx'dy'\varphi(x') K(x'-y')\varphi(y')\right\}\left(\frac{\delta}{\delta\varphi(x)}\exp\left\{i\int dx'\varphi(x')\eta(x')\right\}\right)$$

$$= -i\eta(x)\int D\varphi \exp\left\{\frac{i}{2}\int dx'dy'\varphi(x') K(x'-y')\varphi(y') + i\int dx'\varphi(x')\eta(x')\right\}$$

$$= \frac{-i\eta(x)}{\sqrt{Det\hat{K}}}\exp\left\{\frac{-i}{2}\int dx'dy'\eta(x') K^{-1}(x'-y')\eta(y')\right\} \qquad (3.7)$$

二，*Gauss* 型泛函积分计算举例

1，作为预备，下面列举几个 *Gauss* 型多重积分等式。

《多实变数情况》（ A 为正定对称矩阵）：

$$\begin{cases} \int_{-\infty}^{\infty}\exp\left(-A_{ij}x_i x_j\right)\prod_{i=1}^{n} dx_i = \sqrt{\frac{\pi^n}{\det A}} \\[2ex] \int_{-\infty}^{\infty}\exp\left(-A_{ij}x_i x_j + \alpha_i x_i\right)\prod_{i=1}^{n} dx_i = \sqrt{\frac{\pi^n}{\det A}}\exp\left(\frac{1}{4}\tilde{\alpha}A^{-1}\alpha\right) \end{cases}$$

《多复变数情况》（ A 的实部为正定对称矩阵）：

$$\begin{cases} \int_{-\infty}^{\infty}\exp\left(-Z^+ AZ + \beta^+ Z + Z^+\beta\right)\prod_{i=1}^{n}\frac{d^2 z_i}{\pi} = \frac{1}{\det A}\exp\left(\beta^+ A^{-1}\beta\right), \quad d^2 z_i = dx_i dy_i \\[2ex] \int_{-\infty}^{\infty}\exp\left(-Z^+ AZ - \frac{1}{2}\tilde{Z}BZ - \frac{1}{2}Z^+ CZ^* + \alpha^+ Z + Z^+\beta\right)\prod_{i=1}^{n}\frac{d^2 z_i}{\pi} = \\[2ex] \qquad = \left[\det\begin{pmatrix} B & \tilde{A} \\ A & C \end{pmatrix}\right]\exp\left\{\frac{1}{2}\left(\alpha^+, \tilde{\beta}\right)\begin{pmatrix} B & \tilde{A} \\ A & C \end{pmatrix}^{-1}\begin{pmatrix} \alpha^* \\ \beta \end{pmatrix}\right\}, \quad \tilde{B}=B, \tilde{C}=C \end{cases}$$

《多 *Grassmann* 数情况》：

$$\begin{cases} \displaystyle\int \exp\left(-\frac{1}{2}\tilde{\tau}A\tau\right)\prod_{i=1}^{n}d\tau_i = \sqrt{\det A}\,, \quad \tilde{A}=-A\,, \quad \prod_{i=1}^{n}d\tau_i = d\tau_1\cdots d\tau_n \\[3mm] \displaystyle\int \exp\left(-\frac{1}{2}\tilde{\tau}A\tau+\tilde{\alpha}\tau\right)\prod_{i=1}^{n}d\tau_i = \sqrt{\det A}\,\exp\left(-\frac{1}{2}\tilde{\alpha}A^{-1}\alpha\right) \\[3mm] \displaystyle\int \exp\left(-\tau^{+}A\tau+\sigma^{+}\tau+\tau^{+}\sigma\right)\prod_{i=1}^{n}d\tau_i d\tau_i^{*} = \det A\,\exp\left(\sigma^{+}A^{-1}\sigma\right) \end{cases}$$

$$\int \exp\left(-\tau^{+}A\tau-\frac{1}{2}\tilde{\tau}B\tau-\frac{1}{2}\tau^{+}C\tilde{\tau}^{+}+\tilde{\alpha}\tau+\tau^{+}\beta\right)\prod_{i=1}^{n}d\tau_i d\tau_i^{*} =$$

$$= \left[\det\begin{pmatrix} B & -\tilde{A} \\ A & C \end{pmatrix}\right]^{1/2}\exp\left\{-\frac{1}{2}\left(\tilde{\alpha},-\tilde{\beta}\right)\begin{pmatrix} B & -\tilde{A} \\ A & C \end{pmatrix}^{-1}\begin{pmatrix} \alpha \\ -\beta \end{pmatrix}\right\}$$

2，实标场 *Gauss* 型二次齐次型泛函积分（$x=(\vec{x},x_4=ict)$）

$$\boxed{\int D\varphi(x)\exp\left\{-\frac{1}{2}\int dxdy\,\varphi(x)\hat{K}(x,y)\varphi(y)\right\} = \frac{1}{\sqrt{Det\hat{K}}}} \qquad （3.8\mathrm{a}）$$

这里测度 $D\varphi$ 的选取保证积分等式成立。

证明：从有限维对角看，这里指数上非对角形式的双重求和 $\int dxdy\,\varphi(x)K(x,y)\varphi(y)$ 与对角形式的单重求和 $\int dx\,\varphi(x)K(x,x)\varphi(x)$ 只相差一个对 $\varphi(x)$ 的矩阵变换，或者说对中间矩阵 $\hat{k}$ 的相似变换。两者积分至多相差一个矩阵变换 *Jacobi*，将来和分母上同样因子相消。这由（3.8）式结果行列式 $det\hat{K}=det\left(C^{-1}\hat{K}C\right)$ 也可理解。于是只需要证明对角形式即可，

$$\int D\varphi(x)\exp\left\{-\frac{1}{2}\int dx\,\varphi(x)\hat{K}(x,x)\varphi(x)\right\} = \frac{1}{\sqrt{Det\hat{K}}}$$

将它看成是有限维对角情况向连续无限维的推广。按直接计算办法，

$$左 = \lim_{\substack{\Delta x\to 0 \\ n\to\infty}}\int\prod_{i=1}^{n}\frac{d\varphi(x_i)}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\Delta x\cdot\varphi(x_i)K(x_i,x_i)\cdot\varphi(x_i)\right\}$$

$$= \lim_{\substack{\Delta x\to 0 \\ n\to\infty}}\int\prod_{i=1}^{n}\frac{d\varphi_i}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\varphi_i\left(K_{ii}\Delta x\right)\cdot\varphi_i\right\} = \frac{1}{\sqrt{Det\hat{K}}}$$

最后一步是，针对取极限前对角化有限重积分到取极限后无限重积

分（即路径积分或泛函积分）过渡，采用有限维对角矩阵行列式向无限维算符行列式的转换。此时 Δx 自动含在算符行列式求迹的积分运算之中。此式也常写作

$$\int D\varphi(x)\exp\left\{-\frac{1}{2}\int dx\,\varphi(x)\hat{K}\,\varphi(x)\right\} = \frac{1}{\sqrt{Det\hat{K}}} \qquad （3.8b）$$

总之，在 x 取值分立的有限维情况下，普通函数 $K(x)$ 对其后函数 $\varphi(x)$ 的作用相当于对角矩阵；算符 $\hat{K}$ 对其后函数 $\varphi(x)$ 的作用相当于非对角矩阵；而在函数空间中，算符 $\hat{K}$ 的作用可视作 $\hat{K}\varphi(x)=\int dy K(x,y)\varphi(y)$。

3，实标场 *Gauss* 型非齐次二次型泛函积分

$$\int D\varphi(x)\exp\left\{\int dx\exp\left(-\frac{1}{2}\varphi(x)\hat{K}\,\varphi(x)+J(x)\varphi(x)\right)\right\}$$
$$= \frac{1}{\sqrt{Det\hat{K}}}\exp\left\{\frac{1}{2}\int d(xy)J(x)K^{-1}(x,y)J(y)\right\} \qquad （3.9a）$$

测度 $D\varphi$ 的选取应保证此积分在 $J=0$ 时回归到等式（3.8b）。

证明：这也是有限模情况向连续模情况的推广。作函数平移变换（相应的泛函 *Jacobi* 等于 **1**，说明见结尾处）：

$$\varphi(x) \to \Phi(x) = \varphi(x) - \int d^4x' K^{-1}(x,x')J(x')$$

于是有

$$\frac{-1}{2}\int d^4x\,\Phi(x)\hat{K}\,\Phi(x) =$$
$$= \frac{-1}{2}\int d^4x\left[\varphi(x)-\int d^4x' K^{-1}(x,x')J(x')\right]\hat{K}\left[\varphi(x)-\int d^4x'' K^{-1}(x,x'')J(x'')\right]$$

$$= \frac{-1}{2}\int d^4x\,\varphi(x)\hat{K}\,\varphi(x)+\frac{1}{2}\int d^4(xx')K^{-1}(x,x')J(x')\hat{K}\varphi(x)$$

$$+\frac{1}{2}\int d^4(xx'')\varphi(x)\hat{K}K^{-1}(x,x'')J(x'')$$

$$-\frac{1}{2}\int d^4(xx'x'')K^{-1}(x,x')J(x')\hat{K}K^{-1}(x,x'')J(x'')$$

$$= \frac{-1}{2} \int d^4 x\, \varphi(x) \hat{K} \varphi(x) + \frac{1}{2} \int d^4(xx') K^{-1}(x,x') J(x') \hat{K} \varphi(x)$$

$$+ \frac{1}{2} \int d^4 x\, \varphi(x) J(x) - \frac{1}{2} \int d^4(xx') K^{-1}(x,x') J(x') J(x)$$

如果 $\hat{K}(x)$ 中含 $\partial_\mu^{(x)}$ 二阶偏导数，则第二项经分部积分两次，从而可知第二、第三两项相等。将第四项移至等式左边，即得下面等式：

$$\frac{-1}{2} \int d^4 x\, \varphi \hat{K} \varphi + \int d^4 x\, \varphi J = \frac{-1}{2} \int d^4 x\, \Phi(x) \hat{K} \Phi(x) + \frac{1}{2} \int d^4(xy) J(x) K^{-1}(x,y) J(y)$$

将此式右边替代（3.9a）式左边的指数，并利用（3.8b）式，即得

$$\int D\Phi \exp\left\{ \frac{-1}{2} \int d^4 x\, \Phi(x) \hat{K} \Phi(x) + \frac{1}{2} \int d^4(xy) J(x) K^{-1}(x,y) J(y) \right\}$$

$$= \frac{1}{\sqrt{Det\hat{K}}} \exp\left\{ \frac{1}{2} \int d^4(xy) J(x) K^{-1}(x,y) J(y) \right\} \qquad \text{证毕。}$$

最后，说明此变换的 *Jacobi* 为 **1**。即便对多分量的实标场，由于

$$\frac{\delta \varphi_i(x)}{\delta \Phi_j(y)} = \delta_{ij} \delta(x-y) = \delta_{ij} \langle x | \hat{I} | y \rangle$$

于是 $J\left(\dfrac{\delta \varphi_i}{\delta \Phi_j} \right) = \det \hat{I} = \exp\left\{ tr \ln \hat{I} \right\} = 1$ 。 （3.9a）式也常写作

$$\boxed{\begin{array}{l} \int D\varphi(x) \exp\left\{ \frac{i}{2} \int d(xy) \varphi(x) K(x,y) \varphi(y) + i \int dx\, \varphi(x) J(x) \right\} \\[2mm] = \frac{1}{\sqrt{Det\hat{K}}} \exp\left\{ \frac{-i}{2} \int d(xy) J(x) K^{-1}(x,y) J(y) \right\} \end{array}} \qquad (3.9b)$$

此式的具体算例是：有质量的矢量玻色子场。即令

$$\hat{K}_{\mu\nu}(x) = \left[\delta_{\mu\nu} \left(\partial_\mu^{(x)2} - M^2 \right) - \partial_\mu^{(x)} \partial_\nu^{(x)} \right], \quad M^2 \to M^2(1 - i\varepsilon)$$

$$\boxed{\begin{array}{l} \int \prod_\mu DB_\mu \exp\left\{ i \int d^4 x \left[\frac{1}{2} B_\mu(x) \hat{K}_{\mu\nu}(x) B_\nu(x) + J_\mu(x) B_\mu(x) \right] \right\} \\[2mm] = \frac{1}{\sqrt{|Det\hat{K}_{\mu\nu}|}} \exp\left\{ -\frac{1}{2} \int d^4(xy) J_\mu(x) D_F(x,y)_{\mu\nu} J_\nu(y) \right\} \end{array}} \qquad (3.9c)$$

（**3.9c**）式的测度为 $DB_\mu = D\left(\dfrac{B_\mu}{\sqrt{2\pi i}}\right)$。同时有

$$\hat{K}_{\mu\nu} D_F(x,y)_{\nu\lambda} = i\delta_{\mu\lambda}\delta(x-y), \quad D_F(x,y)_{\nu\lambda} = -i\int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} \frac{(\delta_{\nu\lambda} + k_\nu k_\lambda/k^2)}{\sqrt{Det\hat{K}_{\mu\nu}}}$$

$$\hat{K}_{\mu\nu}\left(\hat{K}\right)^{-1}_{\nu\lambda}(x,y) = \delta_{\mu\lambda}\delta(x-y)$$

4，复标场情况

$$\boxed{\int D\varphi D\varphi^* \exp\left\{-\int dxdy\left[\varphi^+(x)A(x,y)\varphi(y) + \frac{1}{2}\tilde{\varphi}(x)B(x,y)\varphi(y)\right.\right.}$$

$$\left.\left. + \frac{1}{2}\varphi^+(x)C(x,y)\tilde{\varphi}^+(y)\right] + \int dx\left[J^+(x)\varphi(x) + \varphi^+(x)J(x)\right]\right\}$$

$$\boxed{= \left[\det\begin{pmatrix}\hat{B} & \tilde{\hat{A}}\\ \hat{A} & \hat{C}\end{pmatrix}\right]^{-1/2} \exp\left\{\frac{1}{2}\int dxdy\,(J^+(x)\quad \tilde{J}(x))\begin{pmatrix}B(x,y) & \tilde{A}(x,y)\\ A(x,y) & C(x,y)\end{pmatrix}^{-1}\begin{pmatrix}\tilde{J}^+(y)\\ J(y)\end{pmatrix}\right\}}$$

（**3.10a**）

这里测度为 $D\varphi D\varphi^* = \prod_x \dfrac{(\Delta x)^4}{2\pi} d\varphi d\varphi^*$，$\tilde{A}(x,y) = A(y,x)$。

证明：（**3.10a**）式是有限模情况向连续模情况的推广。左边为

$$\int D\varphi D\varphi^* \exp\left\{-\frac{1}{2}\int dxdy\,(\tilde{\varphi}(x)\quad \varphi^+(x))\begin{pmatrix}B(x,y) & A(y,x)\\ A(x,y) & C(x,y)\end{pmatrix}\begin{pmatrix}\varphi(y)\\ \tilde{\varphi}^+(y)\end{pmatrix}\right.$$

$$\left. + \int dx\,(J^+(x)\quad \tilde{J}(x))\begin{pmatrix}\varphi(y)\\ \tilde{\varphi}^+(y)\end{pmatrix}\right\}$$

利用（**3.9b**）式，得

$$左 = \left[\left|\det\begin{pmatrix}B & \tilde{A}\\ A & C\end{pmatrix}\right|\right]^{-1/2} \exp\left\{\frac{1}{2}\int dxdy\,(J^+(x)\quad \tilde{J}(x))\begin{pmatrix}B & \tilde{A}\\ A & C\end{pmatrix}^{-1}(x,y)\begin{pmatrix}\tilde{J}^+(y)\\ J(y)\end{pmatrix}\right\}$$

证毕。

（**3.10a**）式具体算例是复标量场：$\hat{A}(x) = -i\left(\partial^{(x)2}_\mu - m^2\right)$，$m^2 \to m^2 - i\varepsilon$，

$$\int D\varphi D\varphi^* \exp\left\{ i\int d^4 x \left[\varphi^* \partial_\mu^2 \varphi - m^2 \varphi^* \varphi + J^* \varphi + J\varphi^* \right] \right\} =$$
$$= \left| Det\left(\partial_\mu^2 - m^2 \right) \right|^{-1} \exp\left\{ -\int d^4(xy) J^*(x) \Delta_F(x,y) J(y) \right\}$$ （3.10b）

此处测度为 $D\varphi D\varphi^* = \dfrac{D\varphi}{\sqrt{\pi i}} \dfrac{D\varphi^*}{\sqrt{\pi i}}$，并且有

$$\hat{A}(x) A^{-1}(x,y) = \delta(x-y) \rightarrow A^{-1}(x,y) = \Delta_F(x,y)$$

5，*Gauss* 型二次齐次 *Grassmann* 数泛函积分

$$\int D\psi D\bar{\psi} \exp\left\{ -\int dx\, \bar{\psi} K \psi \right\} = DetK$$ （3.11）

证明：这是向连续无穷自由度的推广，可由相应式证明过程得知。

6，*Gauss* 型二次非齐次 *Grassmann* 数泛函积分

设 $A(x)$ 为普通函数，可证有

$$\int D\psi D\bar{\psi} \exp\left\{ \int dx \left[-\bar{\psi} K \psi + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x) \right] \right\} =$$
$$= \left| DetK \right| \exp\left\{ \int dx\, \bar{\eta}(x) K^{-1}(x,x) \eta(x) \right\}$$ （3.12a）

或

$$\int D\psi D\bar{\psi} \exp\left\{ i\int dx \left[-\bar{\psi} K \psi + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x) \right] \right\} =$$
$$= \left| DetK \right| \exp\left\{ -i\int dx\, \bar{\eta}(x) K^{-1}(x,x) \eta(x) \right\}$$ （3.12b）

证明：在函数空间中（取定一组正交完备基，将 ψ 和 $\bar{\psi}$ 展开，知这时是对角矩阵情况），普通函数 $K(x)$ 对应的矩阵是对角的，于是直接使用有限模情况公式即可。

对于 $K(x)$ 是算符的情况，有

$$\int D\psi D\bar{\psi} \exp\left\{ i\int dx \left[-\bar{\psi} \hat{K} \psi + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x) \right] \right\} =$$
$$= \left| Det\hat{K} \right| \exp\left\{ -i\int dxdy\, \bar{\eta}(x) K^{-1}(x,y) \eta(y) \right\}$$ （3.12c）

此 式 具 体 算 例 是 旋 量 场： $\hat{K}(x) = -\left(\gamma_\mu \partial_\mu^{(x)} + \kappa \right)$, $\kappa = m - i\varepsilon$。

$\hat{K}^{-1} = \hat{K}^{-1}(x,y)$ 的定义是：用它右乘 $\hat{K}(x)$ 后等于 $\delta(x-y)$，即

$$\hat{K}(x)\hat{K}^{-1}(x,y) = -\left(\gamma_\mu \partial_\mu^{(x)} + \kappa\right) K^{-1}(x,y) = \delta(x-y)$$

$$\therefore \quad K^{-1}(x,y) = \frac{1}{\hat{K}(x)}\delta(x-y) = \int \frac{d^4 p}{(2\pi)^4} e^{ip(x-y)} \frac{-1}{i\not{p} + m(1-i\varepsilon)} = -iS_F(x,y)$$

证明：这时在函数空间中，$\hat{K}$ 对其后 $\psi(x)$ 的作用相当于非对角矩阵。

此时 $\hat{K}^{-1}$ 定义见前面叙述。证明也直接用有限模公式，

左边＝

$$= \int D\psi D\bar{\psi} \exp\left\{\int d^4 x\left[-\bar{\psi}(x)\left(-i\hat{K}(x)\right)\psi(x) + \left(i\bar{\eta}(x)\right)\psi(x) + \bar{\psi}(x)\left(i\eta(x)\right)\right]\right\}$$

$$= \left|Det\left(-i\hat{K}(x)\right)\right| \exp\left\{\int d^4(xy)\left(i\bar{\eta}(x)\right)\left(-i\hat{K}(x)\right)^{-1}\left(i\eta(y)\right)\right\}$$

$$= \left|Det\hat{K}(x)\right| \exp\left\{-i\int d^4(xy)\bar{\eta}(x)\left(\hat{K}(x)\right)^{-1}\eta(y)\right\}$$

三，泛函 δ – 函数

1，泛函 δ – 函数的两种表达式

i，第一种表达式

$$\boxed{\delta[\Omega] = \prod_x \delta(\Omega(x)) = \int \left[D\xi(x)\right]\exp\left\{i\int d^4 x' \Omega(x')\xi(x')\right\}} \quad （3.13a）$$

或

$$\boxed{\delta[\varphi - \psi] = \prod_x \delta(\varphi(x) - \psi(x)) = \int \left[D\xi(x)\right]\exp\left\{i\int d^4 x'(\varphi(x') - \psi(x'))\xi(x')\right\}}$$

$$（3.13b）$$

并有

$$\boxed{\int D\varphi F[\varphi]\delta[\varphi - \psi] = F[\psi]} \quad （3.14）$$

对 *Gauss* 型泛函 $F[\varphi]$ 情况，（3.14）式可以直接证明如下：

证明：设泛函 $F[\varphi] = \exp\left\{\frac{i}{2}\int d^4(xy)\varphi(x)K(x-y)\varphi(y)\right\}$，将此泛函乘

以泛函 δ – 函数并作泛函积分，为

左边＝

$$= \int D\varphi \exp\left\{\frac{i}{2}\int d^4(xy)\varphi(x)K(x-y)\varphi(y)\right\}\int D\xi \exp\left\{i\int d^4x\left[\varphi(x)-\psi(x)\right]\xi(x)\right\}$$

$$= \int D\xi \exp\left\{-i\int d^4x\,\psi(x)\xi(x)\right\}\int D\varphi \exp\left\{\frac{i}{2}\int d^4(xy)\varphi(x)K(x-y)\varphi(y)+\right.$$
$$\left.+\,i\int d^4x\,\varphi(x)\xi(x)\right\}$$

$$= \int D\xi \exp\left\{-i\int d^4x\,\psi(x)\xi(x)\right\}\frac{1}{\sqrt{\left|Det\hat{K}\right|}}\exp\left\{-\frac{i}{2}\int d^4(xy)\xi(x)K^{-1}(x-y)\xi(y)\right\}$$

$$= \frac{1}{\sqrt{\left|Det\hat{K}\right|}}\int D\xi \exp\left\{\frac{i}{2}\int d^4(xy)\xi(x)\left[-K^{-1}(x-y)\right]\xi(y)+i\int d^4x\left[-\psi(x)\right]\xi(x)\right\}$$

$$= \frac{1}{\sqrt{\left|Det\hat{K}\right|}}\frac{1}{\sqrt{\left|Det\left(-\hat{K}^{-1}\right)\right|}}\exp\left\{\frac{i}{2}\int d^4(xy)\psi(x)K(x-y)\psi(y)\right\}$$

$$= \exp\left\{\frac{i}{2}\int d^4(xy)\psi(x)K(x-y)\psi(y)\right\} = F[\psi] \qquad\qquad 证毕。$$

由此处泛函 δ – 函数定义（**3.13b**）式可知，泛函 δ – 函数为偶函数。

ii, 第二种表达式

$$\boxed{\;\delta\left[F_\alpha[\varphi]\right] \equiv \prod_{x,\alpha}\delta\left(F_\alpha[\varphi]\right) = \lim_{\varepsilon\to 0}\exp\left\{-\frac{i}{2\varepsilon}\sum_\alpha\int d^4x\left(F_\alpha[\varphi]\right)^2\right\}\;} \qquad （3.15）$$

此公式将 *Gauss* 型泛函 δ – 函数表示推广成为 *Fresnel* 型的表示。若分立指标 α 为指定的，则右边指数没有对 α 求和。

证明：首先将普通 δ – 函数的 *Gauss* 型表示推广到 *Fresnel* 型表示；

$$\delta(x) = \lim_{b\to 0}\frac{1}{\sqrt{\pi b}}e^{-x^2/b} \;\rightarrow\; \delta(x) = \lim_{b\to 0}\frac{e^{i\pi/4}}{\sqrt{\pi b}}e^{-i\,x^2/b}$$

于是有

$$\delta\left[F_\alpha[\varphi]\right] \equiv \prod_{x,\alpha}\delta\left(F_\alpha[\varphi]\right) \propto \lim_{b\to 0}\exp\left\{-\frac{i}{b}\sum_{x,\alpha}\left(F_\alpha[\varphi]\right)^2\right\}$$

$$= \lim_{\substack{b \to 0 \\ \Delta x \to 0}} \exp\left\{ -\frac{i}{b(\Delta x)^4} \sum_\alpha \int d^4 x \left(F_\alpha[\varphi]\right)^2 \right\}$$

$$= \lim_{\varepsilon \to 0} \exp\left\{ \frac{-i}{2\varepsilon} \sum_\alpha \int d^4 x \left(F_\alpha[\varphi]\right)^2 \right\}$$

这里在"$\propto$"号的一步中略去了 δ – 函数中与函数 $\varphi(x)$ 无关的常数因子，因为它与生成泛函分母归一化系数中的对应因子相消。

2，泛函 δ – 函数的自变量变换

i, 泛函 δ – 函数为偶函数。即总有

$$\boxed{\delta[\varphi - \psi] = \delta[\psi - \varphi]} \tag{3.16}$$

由第一种表达式（**3.13b**）可知，这相当于对泛函积分变数做变换 $\xi(x) \to -\xi(x)$ 时，积分数值不变。这只需注意，做此变换时泛函积分上下限也变换，所以有 $\int[D\xi(x)] = \int[D(-\xi(x))]$。

ii, 泛函 δ – 函数的自变量变换

$$\boxed{\delta\left[\varphi - \hat{M}\psi\right] = \left|Det\hat{M}\right|^{-1} \delta\left[\psi - \hat{M}^{-1}\varphi\right]} \tag{3.17}$$

这里 $\delta\left[\varphi - \hat{M}\psi\right] = \prod_x \delta\left(\varphi(x) - \left(\hat{M}\psi\right)(x)\right)$。

证明：有映射 $\hat{M}: \psi(x) \to \varphi(x)$（即 $\hat{M}\psi = \varphi$），$F[\varphi] = F[\varphi[\psi]] \equiv G[\psi]$。设 $\tilde{\varphi}$ 和 $\tilde{\psi}$ 为满足 $\hat{M}\tilde{\psi} = \tilde{\varphi}$ 的任意一组函数对，按泛函 δ – 函数定义有

$$\begin{cases} \int D\varphi \cdot \delta\left[\varphi - \hat{M}\tilde{\psi}\right] F[\varphi] = F\left[\hat{M}\tilde{\psi}\right] \\ \int D\psi \cdot \delta\left[\psi - \hat{M}^{-1}\tilde{\varphi}\right] G[\psi] = G\left[\hat{M}^{-1}\tilde{\varphi}\right] \end{cases}$$

于是第一式的右边和左边分别等于

$$\begin{cases} F\left[\hat{M}\tilde{\psi}\right] = F[\tilde{\varphi}] = G[\tilde{\psi}] = G\left[\hat{M}^{-1}\tilde{\varphi}\right] \\ \int D\psi \left|Det\dfrac{\delta\varphi}{\delta\psi}\right| \delta\left[\varphi - \hat{M}\tilde{\psi}\right] F[\varphi[\psi]] = \int D\psi \left|Det\hat{M}\right| \delta\left[\varphi - \hat{M}\tilde{\psi}\right] G[\psi] \end{cases}$$

与第二式相比较，并略去"$\tilde{\ }$"，即得（**3.17**）式。 证毕。

于是，$\delta-$函数的前面定义又可以改写为：

$$\delta\left[\varphi-\hat{M}\psi\right]=\prod_x\delta\left(\varphi(x)-\int dy M(x-y)\psi(y)\right)=\left|Det\hat{M}\right|^{-1}\delta\left[\psi-\hat{M}^{-1}\varphi\right]$$

$$=\left|Det\hat{M}\right|^{-1}\prod_x\delta\left(\psi(x)-\int dy M^{-1}(x-y)\varphi(y)\right)$$

$$=\left|Det\hat{M}\right|^{-1}\int D\xi\exp\left\{i\int d^4x\left[\int M^{-1}(x-y)\varphi(y)dy-\psi(x)\right]\xi(x)\right\}$$

（**3.18**）

这些记法逻辑上是自洽的。因为对任意泛函 $F[\varphi]$，按 $\delta-$ 函数定义：

$$\int D\varphi F[\varphi]\delta\left[\varphi-\hat{M}\psi\right]=F\left[\hat{M}\psi\right]$$

另一方面，按泛函积分变换和 $\delta-$ 函数变换的定义，上式左边为：

$$\int D\psi\left|Det\frac{\delta\varphi}{\delta\psi}\right|F[\varphi[\psi]]\delta\left[\varphi-\hat{M}\psi\right]$$

$$=\int D\psi\left|Det\hat{M}\right|F[\varphi[\psi]]\delta\left[\varphi-\hat{M}\psi\right]$$

$$=\int D\psi F[\varphi[\psi]]\delta\left[\psi-\hat{M}^{-1}\varphi\right]=F\left[\varphi\left[\hat{M}^{-1}\varphi\right]\right]$$

$$=F\left[\hat{M}\hat{M}^{-1}\varphi\right]=F[\varphi]=F\left[\hat{M}\psi\right]$$

四，泛函 *Fourier* 变换

1，泛函 *Fourier* 变换定义

显然，下面 *Gauss* 型泛函积分公式，

$$\boxed{\begin{aligned}Z[\eta]&=\int D\varphi\exp\left\{\frac{i}{2}\int dxdy\,\varphi(x)K(x-y)\varphi(y)+i\int dx\,\varphi(x)\eta(x)\right\}\\&=\frac{1}{\sqrt{Det\hat{K}}}\exp\left\{\frac{-i}{2}\int dxdy\,\eta(x)K^{-1}(x-y)\eta(y)\right\}\end{aligned}}$$

（**3.19**）

是一个**泛函 *Fourier* 变换式**：中间的指数二次型的被积泛函 $F[\varphi]$，

$$F[\varphi]=\exp\left\{\frac{i}{2}\int dxdy\,\varphi(x)K(x-y)\varphi(y)\right\}$$

是场量函数 $\varphi(x)$ 的 *Gauss* 型泛函 $F[\varphi]$，是此**泛函 *Fourier* 变换的原泛函**；右边最后积出的结果是外源函数 $\eta(x)$ 的 *Gauss* 型泛函 $Z[\eta]$，

是**泛函 *Fourier* 变换的像泛函**。就是说，这两个泛函之间的关系为：

$$\text{“\textbf{泛函} } Z[\eta] \text{ \textbf{是泛函} } F[\varphi] \text{ \textbf{的} \textit{Fourier} \textbf{变换}。”}$$

更一般的说，称：

$$\text{“} \int D\varphi\, \varphi(x_1)\cdots\varphi(x_n)\exp\left\{\frac{i}{2}\int dxdy\, \varphi(x)K(x-y)\varphi(y)+i\int dx\, \varphi(x)\eta(x)\right\}$$

$$=\frac{1}{\sqrt{Det\hat{K}}}\frac{\delta}{i\delta\eta(x_1)}\cdots\frac{\delta}{i\delta\eta(x_n)}\exp\left\{\frac{-i}{2}\int dxdy\, \eta(x)K^{-1}(x-y)\eta(y)\right\} \quad \textbf{(3.20a)}$$

这个关于外源 $\eta(x)$ 的 *Gauss* 型泛函是下面关于场量函数 $\varphi(x)$ 的

***Gauss* 型泛函**

$$F[\varphi]=\varphi(x_1)\cdots\varphi(x_n)\exp\left\{\frac{i}{2}\int dxdy\, \varphi(x)K(x-y)\varphi(y)\right\} \quad \textbf{(3.20b)}$$

的 *Fourier* 变换。”

一般说，[**泛函 *Fourier* 变换定义**]：

"**如果下面泛函积分 $Z[\eta]$ 存在，**

$$Z[\eta]=\int D\varphi(x)\, F[\varphi(x)]\exp\left\{i\int dx'\, \varphi(x')\eta(x')\right\}$$

称外源泛函 $Z[\eta]$ 为场量泛函 $F[\varphi]$ 的 *Fourier* 变换像泛函。”

2，例算。作为泛函 *Fourier* 变换和 δ – 函数的一个应用，求证如下结论： 在变数变换

$$\varphi(x)=\hat{M}\psi(x)=\int M(x,y)\psi(y)dy$$
$$=c_0(x)+\psi(x)+\int N(x,y)\psi(y)dy$$
$$\equiv c_0(x)+\psi(x)+N[\psi]$$

之下，有下述泛函积分的转换

$$\int D\varphi \exp\left\{\frac{i}{2}\int \varphi(x)K(x-y)\varphi(y)dxdy + i\int dx\varphi(x)J(x)\right\} =$$

$$= \int D\psi\, Det\left(1+\frac{\delta N}{\delta\psi}\right)\exp\left\{\frac{i}{2}\int(\hat{M}\psi)(x)K(x-y)(\hat{M}\psi)(y)dxdy\right. \quad (3.21)$$

$$\left. + i\int(\hat{M}\psi)(x)J(x)dx\right\}$$

证明：变换前的被积函数是 *Gauss* 型，积分有定义。根据 *Fourier* 变换思想只需证明 *Fourier* 变换等式（3.21）两边的原函数相等即可。

显然，等式左边表达式（记作 $I[J]$）表明，其原函数 $\tilde{I}[\varphi]$ 是

$$\exp\left\{\frac{i}{2}\int\varphi(x)K(x-y)\varphi(y)dxdy\right\}$$

而等式右边表达式（记作 $I'[J]$）的原函数 $\tilde{I}'[\psi]$ 是

$$\tilde{I}'[\psi] = \int DJ\cdot I'[J]\cdot\exp\left\{-i\int dxJ(x)\psi(x)\right\}$$

$$= \int D\psi\int DJ\, Det\left(1+\frac{\delta N}{\delta\psi}\right)\exp\left\{\frac{i}{2}\int(\hat{M}\psi)(x)K(x-y)(\hat{M}\psi)(y)dxdy\right\}\cdot$$

$$\cdot\exp\left\{i\int J(x)\left[(\hat{M}\psi)(x)-\psi(x)\right]dx\right\}$$

$$= \int D\psi\, Det\left(1+\frac{\delta N}{\delta\psi}\right)\exp\left\{\frac{i}{2}\int(\hat{M}\psi)(x)K(x-y)(\hat{M}\psi)(y)dxdy\right\}\cdot$$

$$\cdot\delta\left[(\hat{M}\psi)(x)-\psi(x)\right]$$

$$= \int D\psi\, Det\left(1+\frac{\delta N}{\delta\psi}\right)\exp\left\{\frac{i}{2}\int\psi(x)K(x-y)\psi(y)dxdy\right\}\cdot$$

$$\cdot\delta\left[(\hat{M}\psi)(x)-\psi(x)\right]$$

$$= \int D\varphi\exp\left\{\frac{i}{2}\int\psi(x)K(x-y)\psi(y)dxdy\right\}\delta\left[\varphi(x)-\psi(x)\right]$$

$$= \exp\left\{\frac{i}{2}\int\kappa(x)K(x-y)\varphi(y)dxdy\right\}$$

五，泛函积分的变数变换与泛函 *Jacobi*

1，泛函积分的变数变换，泛函 *Jacobi*

设线性函数映射 $M: \varphi(x) \to \psi(x)$，老场量 $\varphi(x)$ 由新场量 $\psi(x)$ 按如下变换相互联系

$$\varphi_\alpha(x) = \int M_{\alpha\beta}(x,y)\psi_\beta(y)dy \equiv \left(\hat{M}\psi(x)\right)_\alpha \qquad (3.22a)$$

α, β 为内禀、外在全部附属空间的分量指标。在此泛函变数变换下，泛函积分变换为

$$\int F\left[\varphi_\alpha(x)\right]\prod_\alpha D\varphi_\alpha(x) = \int F\left[\psi_\beta(y)\right]J\left(\frac{\delta\varphi_\alpha}{\delta\psi_\beta}\right)\prod_\beta D\psi_\beta(y)$$

积分测度的相应变换为

$$D\varphi(x) = \prod_{\alpha,x}\frac{d\varphi_\alpha(x)}{\sqrt{C}} = \prod_{\alpha,x}\frac{\delta\varphi_\alpha(x)}{\delta\psi_\beta(x)}\cdot\prod_{\beta,x}\frac{d\psi_\beta(x)}{\sqrt{C}}$$

$$\therefore \qquad \boxed{D\varphi(x) = \left|Det\hat{M}\right|D\psi(x)} \qquad (3.22b)$$

$$\int F\left[\varphi_\alpha(x)\right]\prod_\alpha D\varphi_\alpha(x) = \int F\left[\psi_\beta(y)\right]J\left(\frac{\delta\varphi_\alpha}{\delta\psi_\beta}\right)\prod_\beta D\psi_\beta(y)$$

最后一步等号是因为

$$\left(\frac{\delta\varphi_\alpha(x)}{\delta\psi_\beta(y)}\right) = M(x,y) \to$$

$$\prod_x\left(\frac{\delta\varphi_\alpha(x)}{\delta\psi_\beta(x)}\right) = \exp\left\{\int dx\ln\left(\frac{\delta\varphi_\alpha(x)}{\delta\psi_\beta(x)}\right)\right\} = \exp\left\{\int dxdy\delta(x-y)\ln M(x,y)\right\}$$

$$= \exp\left\{tr\ln\hat{M}\right\} = Det\hat{M}$$

于是，泛函 *Jacobi* 为

$$\boxed{J\left(\frac{\delta\varphi_\alpha}{\delta\varphi_\beta'}\right) = \begin{cases} Det\hat{M} = \exp\left[tr\left(\ln\hat{M}\right)\right], & for\ boson \\ \left(Det\hat{M}\right)^{-1} = \exp\left[-tr\left(\ln\hat{M}\right)\right], & for\ fermion \end{cases}} \qquad (3.23)$$

总之，在此变数变换下，泛函积分的表达式改换为

$$\boxed{Z[\eta] = \int D\varphi\, F[\varphi] = \int D\psi\left|Det\hat{M}\right|F\left[\hat{M}\psi\right]} \qquad (3.24)$$

这里 $\varphi(x)$ 与 $\psi(x)$ 的关系由（**3.22a**）式决定。

2，无穷小泛函变分，泛函 *Jacobi*

如果场变数的变化是无穷小的泛函变分，即变换 $\hat{M}$ 接近于恒等变换 $\hat{M} = \hat{I} + \hat{A}$，$\hat{A}$ 为含有穷小常数的算符，如同经常做的那样。这时可对上面 *Jacobi* 作近似，从而写作更方便的形式：

$$\exp\left[\pm tr\left(\ln\hat{M}\right)\right] = \exp\left[\pm tr\left(\ln\left(\hat{I} + \hat{A}\right)\right)\right] \Box \exp\left[\pm tr\hat{A}\right] \Box \hat{I} \pm tr\hat{A}$$

最后可得，此无穷小变换的泛函 *Jacobi* 为

$$\boxed{J\left(\frac{\delta\varphi_\alpha}{\delta\psi_\beta}\right) \Box \begin{cases} \hat{I} + tr\hat{A}, & for\ boson \\ \hat{I} - tr\hat{A}, & for\ fermion \end{cases}} \tag{3.25}$$

举例：泛函 $\int D\psi\, D\bar{\psi} \prod_\mu DB_\mu\, F\left(\psi,\bar{\psi},B_\mu\right)$ 经受如下场量变换

$$\begin{cases} \psi(x) \to \psi'(x) = \left(1 + \frac{3}{2}\lambda(x)\right)\psi(x) \\ \bar{\psi}(x) \to \bar{\psi}'(x) = \left(1 + \frac{3}{2}\lambda(x)\right)\bar{\psi}(x) \\ B_\mu(x) \to B'_\mu(x) = \left(1 + \lambda(x)\right)B_\mu(x) \end{cases} \tag{3.26a}$$

这时，泛函 *Jacobi* 为

$$J(\lambda) = J_\psi(\lambda)\, J_{\bar{\psi}}(\lambda)\, J_B(\lambda)$$

于是，比如对 ψ，有

$$\psi'(x) = \int \delta(x-y)\left(1 + \frac{3}{2}\lambda(x)\right)\psi(y)\, d^4y$$

$$\therefore \quad \langle x|\hat{M}|y\rangle = M(x,y) = \delta(x-y)\left(1 + \frac{3}{2}\lambda(x)\right)$$

因此

$$\begin{cases} \langle x|\hat{M}_\psi - 1|y\rangle = \frac{3}{2}\lambda(x)\delta(x-y), & for\ \psi - 4\dim \\ \langle x|\hat{M}_{\bar{\psi}} - 1|y\rangle = \frac{3}{2}\lambda(x)\delta(x-y), & for\ \bar{\psi} - 4\dim \\ \langle x|\hat{M}_B - 1|y\rangle = \lambda(x)\delta(x-y), & for\ B_\mu - 4\dim \end{cases}$$

$$\begin{cases} tr\left(\hat{M}_{\psi}-1\right) = 4\cdot\dfrac{3}{2}\delta\left(0\right)\int \lambda\left(x\right)d^4x = 6\delta\left(0\right)\int \lambda\left(x\right)d^4x \\[2mm] tr\left(\hat{M}_{\bar{\psi}}-1\right) = 4\cdot\dfrac{3}{2}\delta\left(0\right)\int \lambda\left(x\right)d^4x = 6\delta\left(0\right)\int \lambda\left(x\right)d^4x \\[2mm] tr\left(\hat{M}_{B}-1\right) = 4\cdot\delta\left(0\right)\int \lambda\left(x\right)d^4x = 4\delta\left(0\right)\int \lambda\left(x\right)d^4x \end{cases}$$

最后得（加减号按 *boson* 或 *fermion* 而定）

$$J\left(\lambda\right) = \left(1-6\delta\left(0\right)\int \lambda\left(x\right)d^4x\right)\left(1-6\delta\left(0\right)\int \lambda\left(x\right)d^4x\right)\left(1+4\delta\left(0\right)\int \lambda\left(x\right)d^4x\right)$$

$$\Box\ 1-8\delta\left(0\right)\int \lambda\left(x\right)d^4x$$

若对 $\lambda(x)$ 求泛函导数（这是常常需要的），则有

$$\boxed{\dfrac{\delta J\left(\omega\right)}{\delta\lambda\left(y\right)} = -8\delta\left(0\right)}$$
（3.26b）